Ramanujan Mathematical Society

LITTLE MATHEMATICAL TREASURES

VOLUME 1

A Gateway to

MODERN MATHEMATICS

Adventures in Iteration I

SHAILESH A SHIRALI

Universities Press

Ramanujan Mathematical Society

LITTLE MATHEMATICAL TREASURES

VOLUME 1

A Gateway to MODERN MATHEMATICS

Adventures in Iteration I

SHAILESH A SHIRALI

Universities Press

A Gateway to

MODERN MATHEMATICS

Adventures in Iteration I

Ramanujan Mathematical Society
Little Mathematical Treasures

Editorial Policy

In times past, mathematics was studied and researched by a select few—an elite class who valued the subject for its intellectual elegance and perhaps also for its philosophic connotations. But today it is studied and practised by an extremely large number of individuals, for mathematics has acquired a pivotal role in the sustenance of a technologically advanced society.

Broadly speaking, there are two distinct groups of students to whom expository material in mathematics may be addressed:

a. students who have a deep interest in mathematics and wish to pursue a career involving research and teaching of mathematics;

b. students who are good at mathematics but wish to pursue a career in some other discipline.

Clearly such students require suitable material (in addition to contact with inspiring teachers and practising mathematicians) to nurture their talent in the subject.

To this list we may add professionals in industry and government and teachers in mathematics and other disciplines who require a good understanding of certain areas of mathematics for their professional work.

The proposed series is addressed to mathematically mature readers and to bright students in their last two years of school education. It is envisaged that the books will contain expository material not generally included in standard school or college texts. New developments in mathematics will be presented attractively using mathematical tools familiar at the high school and undergraduate levels. There will be problem sets scattered through the texts which will serve to draw the reader into a closer hands-on study of the

subject. Readers will be invited to grapple with the subject, and so experience the creative joy of discovery and contact with beauty.

A thing of beauty is a joy forever So it is with mathematics. The discoveries of Archimedes, Apollonius and Diophantus have been sources of joy from very ancient times. It is our hope that these books will serve our readers in a similar manner.

Proposals for publication of a book in the series is invited with a draft of the proposed book. The decision of the Editorial Board will be final.

Ramanujan Mathematical Society

LITTLE MATHEMATICAL TREASURES

VOLUME 1

A Gateway to

MODERN MATHEMATICS

Adventures in Iteration I

SHAILESH A SHIRALI
Consultant, Rishi Valley School
Andhra Pradesh

Universities Press

Universities Press (India) Private Limited

Registered Office
3-6-747/1/A & 3-6-754/1, Himayatnagar, Hyderabad 500 029 (A.P.), INDIA
e-mail: info@universitiespress.com

Distributed by
Orient Blackswan Private Limited

Registered Office
3-6-752 Himayatnagar, Hyderabad 500 029 (A.P.), INDIA
e-mail: info@orientblackswan.com

Other Offices
Bangalore, Bhopal, Bhubaneshwar, Chennai,
Ernakulam, Guwahati, Hyderabad, Jaipur, Kolkata,
Lucknow, Mumbai, New Delhi, Noida, Patna

First published 2009
Reprinted 2010

eISBN 978-81-7371-960-8

e-edition:First Published 2015

ePUB Conversion: Techastra Solutions Pvt. Ltd.

Contents

Preface

This book is about *iterative processes*, i.e., processes that are done repetitively ("reiterated"); such processes often result in sequences which are rich in patterns, making them ideal for investigation. It is written for the student who loves to tinker around with numbers and equations, and with mathematical ideas in general; tinker around just as a child, does when he or she wanders into a wooded region and begins to explore the many marvels and mysteries that are to be found in undisturbed places.

The world of mathematics too is full of wonders but, sadly, many children never see them at all, or even suspect that they exist. The fact that mathematics is a living science and that one can make discoveries and actually contribute to the growth of the subject is hardly ever pointed out at the school level. And yet, the actual fact is that mathematics is *alive*, and not merely a collection of formulas and theorems dating from Greek and Vedic times. New discoveries are constantly being made in the subject every year; new areas of research develop constantly. Problems dating from many centuries get solved; alongside this, new questions get asked, new problems posed. The culture of the subject is such that mathematicians can speak to one another across barriers of space and time—not just decades, but centuries—with ease and familiarity, with no fear of not being understood. To give just a few instances: in the last decade of the twentieth century, the Princeton mathematician Andrew Wiles dramatically laid to rest a question that had been lying unresolved (not for want of trying!) for more than three hundred years—the famous Fermat conjecture:

> *No cube can be written as the sum of two smaller cubes, no fourth power can be written as the sum of two smaller fourth powers, and so on for all higher powers.*

In the last decade of the eighteenth century, Karl F.Gauss (1779–1855), regarded as one of the greatest mathematicians of all time, alongside Archimedes (3rd century BC) and Isaac Newton (1642–1722), then but a

teenager, announced a solution to a question that had been asked more than two thousand years earlier by the Greeks, concerning the possibility of a ruler-and-compass construction of a regular polygon with a specified number of sides. (Gauss answered the question completely: he gave a criterion for finding the numbers n for which a regular n-sided polygon is constructive.) The famous four-colour theorem, which states that every map can be properly coloured using no more than four colours was first conjectured in the 1860s (by a lawyer!) and finally proved in 1976, by Kenneth Appel and Wolfgang Haken: a culmination of more than a century of effort by many generations of mathematicians. The notebooks of the enigmatic mathematical genius from India, Srinivasa Ramanujan (1887–1920), are filled with tantalizing statements that continue to hold deep interest today for a large number of working mathematicians.

Mathematics is, like the air, free for the taking. Just as no one owns a sunset —it is there for everyone to see—so it is with mathematics: it is there for the enjoyment of all. Can this feeling be experienced by those who come into contact with the subject? This is the essential spirit embodied in this book.

****

This book is the first of a two-volume work on the subject of iterations. It is set at an elementary level, and the background knowledge required is modest,: an understanding of elementary algebra, arithmetic and geometry (briefly, the topics covered in most schools in grades 8–10), and an ease in dealing with numbers. Readers may need to refresh their memory of the technique used for computing the greatest common divisor ("GCD") of two given integers. The focus in general is on the discovery of patterns rather than on proofs and derivations. (Those who progress from Volume I to Volume II will find that it is set at a higher level, with formal proof receiving its due place, and substantial usage being made of calculus and complex numbers.)

The book will be found useful by students who are preparing for the Regional Mathematics Olympaid (RMO) and the Indian National Mathematics Olympiad (INMO). It is suitable for self-study by students in the age range 13–18 years, and by lay readers, who will enjoy learning about a topic of great current interest. It can also be used by teachers in a school mathematics club.

The book presupposes a certain degree of mathematical maturity on the part of the reader—an elusive quality that means, roughly, the ability to grasp mathematical concepts and to *think* mathematically. It is the earnest and committed belief of this author that working through this book will itself greatly enhance the young learner's mathematical maturity.

Many of the exercises involve computation, and it would help greatly if the student had access to a computer. Many students today are familiar with programming in *PASCAL,* C+ + and *BASIC* (probably the easiest language to learn and use, though it has been criticized for encouraging poor programming style!). Writing programs to solve the exercises should prove a simple task. (Indeed, many of the exercises provide excellent practice material for the writing of programs.)

Above all, the book is about having fun with figures. Enjoy yourself!

Preview of book

Iterations are introduced in an informal manner in Chapter 1, and this is followed in Chapter 2 by a list of all technical terms used in the book. Chapter 3 presents some illustrative examples. Chapters 4-12 offer numerous iterations from arithmetic and algebra, such as the well known four-numbers iteration, the Kaprekar iteration, and so on. In Chapter 13, the focus shifts to Euclidean geometry, Chapter 14 discusses techniques of proof in the context of iterations, and Chapter 15 presents a possible path along which iterations present themselves very naturally (in the context of solution of equations). In Chapter 16, the material is placed in a graphical framework ("cobwebbing"), and this sheds light on some empirical discoveries made earlier. Chapters 17 and 18 contain problems which the reader is invited to tackle.

Appendix A gives a short reading list, and Appendix B gives complete solutions to all the problems in the book.

Acknowledgements

Grateful acknowledgements of thanks are due, for the assistance that I received from numerous individuals: to Professor Phoolan Prasad, Dr. Jayant Kirtane and Mr. Siddhartha Menon, for going through early versions of the

book and offering helpful and critical comments; to Mr. and Mrs. Mohan Subramaniam, for their help, generously given, in procuring the LaTeX Software; to Mr. K Seshadri, for technical help with the computer systems; to Professor E Sampathkumar for his enthusiastic support in publishing this book; to my friends in the "Olympiad cell," C R Pranesachar ("CRP"), B J Venkatachala ("BJV") and C S Yogananda ("CSY"), with whom I have had numerous iterated adventures; and to Padmapriya, my wife, for her iterated support.

Dedication

I dedicate this book to my ε, Karunya, to her *Ma* Padmapriya, and to my parents, Ashok and Lata Shirali.

Chapter 1

What Are Iterations?

1.1 A card trick

From a card pack remove any twenty-one cards. Ask a friend to mentally select any one of these cards. Announce to your friend that you plan to find out the selected card.

Call the selected card '*X*' for short. Place the cards, face up, in an array measuring 7 × 3, in the order shown by the numbers below (the cards are laid out in the array row by row; the column numbers have been shown in the top row):

I	*II*	*III*
1	2	3
4	5	6
7	8	9
...	...	...
19	20	21

Ask your friend to tell you the column in which X lies. Next, gather up the cards, column by column, so that the column containing X is the *second* one to be picked up. For instance, if X were at position number 7, your friend would say 'column *I*'. You would then pick up the columns in the order (II,I,III) or (*III*, I,II) .

Now repeat the whole process: lay out the cards, row-wise, in the order indicated above: row number 1, then row number 2, …, till row number 7. Once again ask your friend to tell you the column containing X, then pick up the cards column by column just as you did earlier.

Lay out the cards for a third time, do the gathering up yet again, and then lay out the cards once more (row-wise)—this would be the *fourth* time you are doing so. And now—here is the surprise!—you will find that X is in position number 11. There is no uncertainty about it; it will always end up in that position. Now all you have to do is say "Abracadabra" or something like that, and produce the card!

This is a neat card trick to play, and you may enjoy figuring out how it works. Unlike most such tricks, no sleight of hand is involved: palming the cards, hiding them up your sleeve, and so on. Other than putting on a "magician's voice", very little skill is needed to perform the trick.

If you enjoy experimentation, you may want to investigate what role the pair of numbers 3 and 7 play in this trick. What if we used, instead, the numbers 5 and 7? Or 5 and 6? Or 4 and 5? Or 3 and 4? In what way does the result change if we alter the numbers?—and does the trick work at all?

Observe that what you are doing as you perform this trick is to repeat a certain process several times: you start with some arrangement of cards, "do" something to it, do it again, and so on. After four such cycles, the card ends up in some fixed position. The 'answer' comes out the same, no matter how you started out.

We plan now to do a study of such processes, in which some fixed action is repeated several times over. Such processes are known as **ITERATIONS.** If you look up a dictionary, you will find that the word *reiterate* means 'to repeat something over again'. That is exactly what we are doing here.

1.2 The riffle shuffle

Here is another illustration of an iteration that uses a card pack; it is based on a technique of shuffling known as the *riffle shuffle*. The purpose of any shuffle, clearly, is to mix up the cards thoroughly. In a riffle shuffle, well known to professional card players, the shuffler divides the card pack into two halves, holds one half in each hand, then 'drops' the cards alternately, one at a time, from the bottoms of the two packs; one card from the left hand, then one from the right hand, then again one of the left hand, and so on, till the last card. Professionals can execute the shuffle at bewildering speed, making for a very attractive effect.

To illustrate its action, assume that the card pack has ten cards, numbered 1, 2, 3, …, 10, and arranged in this order (reading from top to bottom). A riffle shuffle would place them in the following order:

l, 6, 2, 7, 3, 8, 4, 9, 5, 10.

To see why, it helps to write the numbers 1, 2, 3, …, 10 in the following manner:

Left	Right
1	6
2	7
3	8
4	9
5	10

If we follow this with a second riffle shuffle, we obtain the following arrangement:

1, 8, 6, 4, 2, 9, 7, 5, 3, 10,

and a third shuffle produces:

1, 9, 8, 7, 6, 5, 4, 3, 2, 10.

The shuffles can be continued, and something curious happens if we do so. Perhaps you could do the shuffles on your own and find out what happens. Magicians are familiar with such curiosities and sometimes base their card tricks on them.

1.3 Feedback

An ‘iteration’ in mathematics refers to a process that is done over and over again; each time the process is done, the ‘output’ or result is fed back to the starting point. This ‘feeding back’ is the crucial part, and every iteration has this ingredient in it. A good visual image is provided by the laboratory process of purification by distillation: the distilled liquid is put through the distillation process again and again so as to increase or decrease the concentration of some particular component.

Here is a more familiar example, involving sound. You must have heard, at

some time or the other, the painful howl of the sound system during a music performance. What happens is that sound from the speakers is picked up ('heard') by the microphone, magnified by the amplifier, emitted again from the speakers, picked up yet again by the microphone, …. This cycle happens repeatedly, and the result is the howl.

Chapter 2

Terms and Symbols

As you read through this book, you will encounter many new terms and symbols. A list of such words has been given below, along with their meanings. Please look up the list whenever you come across an unfamiliar word.

Function A function is essentially a machine that converts an *input* (that which goes 'into' the machine) into an *output* (that which comes 'out' of the machine). The following sketch conveys the idea pictorially:

$$\text{INPUT} \rightarrow \boxed{\text{FUNCTION}} \rightarrow \text{OUTPUT.}$$

In the most familiar examples, the input and output are both numbers. In the microphone example mentioned above, the input and output are both sounds.

We often use symbols such as f,g,h,… to denote functions. To denote the output of a function f from an input of x we use the symbol f(x). We depict this visually as follows:

$$x \longrightarrow \boxed{f} \longrightarrow f(x).$$

The symbol ' f(x)' is pronounced ' f of x′.

Example 1. The function f that squares whatever input it receives is written symbolically as f(x) = x2; so f(5) = 25,f(7) = 49,…. Visually we may write:

$$x \longrightarrow \boxed{f} \longrightarrow x^2.$$

Another symbolic form that we shall often use is : f : x↦x2; here the symbol

'↦' (pronounced "maps to") indicates that f "maps" x to x2.

Domain of a function The set of allowable values that can be taken by the input is the *domain* of the function. The domain of a function f is typically denoted by the symbol domain (f), or simply by dom (f).

The domain can be any arbitrary set; the only requirement is that the function should be able to act meaningfully on every element in the set.

Example 2. Consider the reciprocal function R(x) which takes x to 1⁄x. (This is written as: R(x) = 1⁄x, or as R : x↦1⁄x.) Here we exclude 0 from the domain of R, because 1/0 is not defined.

Example 3. Consider the function which takes x to log2x. Here we exclude from the domain all numbers less than or equal to 0, because log2x is not defined for $x \leq 0$. (The base could be 10 instead of 2, or for that matter any positive number not equal to 1.)

Range of a Function The set of all possible output values that can produced by the function, when the input takes every possible value in the domain, is the *range* of the function.

Example 4. For positive integers n, let f(n) denote the units digit of n2, e.g., f(12) = 4, since 122 = 144. (All references are to base-10.) The domain of f is the set of positive integers N, and the range is the set

$$\{0,1,4,5,6,9\}.$$

Example 5. For positive integers n, let g(n) denote the units digit of 3n,e. g ., g(6) = 9 , since 36 = 729. The domain of g is N, as earlier, and the range is the set

$$\{1, 3, 7, 9\}.$$

Example 6. Consider the function R(x) given by R(x) = 1⁄x. Let domain (f) be the set of real numbers between 1 and 2; the usual notation for this set is [1, 2]. Then the range is the set of real number between 1/2 and 1, denoted by ⌊1⁄2,1] .

Iteration, Iterative Process Any action which is performed repeatedly will

be referred to as an *iteration*. The action itself is called the *iterative step*. The specific rule used to do the iteration is the *iterative function*. (Later we shall suppress the word 'iterative' and simply say 'function'.)

Sometimes the action is also referred to as a "*mapping*" or a "*transformation*".

input That which goes "into" each iterative step is called the *input*. (If $\mapsto f(x)$, then x is the input)

output That which comes "out" of each iterative step is called the *output*. (If $x \mapsto f(x)$, then f(x) is the output.)

Example 7. For the riffle shuffle example presented earlier, the iterative function is the riffle shuffle itself, and for the input

$$(1, 6, 2, 7, 3, 8, 4, 9, 5, 10),$$

the output is

$$(1, 8, 6, 4, 2, 9, 7, 5, 3, 10).$$

Example 8. Suppose the iterative step is: *add 2 to the input*. The function is in this case described symbolically by:

$$f(x) = x + 2.$$

Here an input of 4 produces an output of 6, an input of 10 produces an output of 12, and so on.

Seed Value The input value with which we start the iteration is the *seed* of the iteration.

Iterative Sequence, Orbit The sequence which is generated by an iterative function together with a seed is an *iteration sequence*. Other words used to describe an iterative sequence are *orbit* and *trajectory*.

Example 9. For the iterative function $f(x) = x + 2$, the iterative sequence produced by the seed $x = 5$ is

$$5, 7, 9, 11, 13, 15, 17, 19, 21, 23, ...$$

Example 10. For the function f(x) = x3, the iterative sequence produced by the seed value x = 3 is

3, 27, 19683, 7625597484987,

443426488243037769948249630619149892803, ….

(Here 33 = 27,273 = 19683, 196833 =7625597484987, and so on.)

Example 11. For the function f(x) = 1 - x, the orbit of the seed x = 0 is

0,1,0,1,0,1,0,1,….

Notation for Sequences, Orbits Sequences are often placed within angular brackets ('⟨' and '⟩'). For example, we write:

⟨5,7,9,11,13,15,17,19,…⟩.

The orbit of the seed number x produced by the function f is usually shown in the following manner:

⟨x,f(x),f(f(x)),f(f(f(x))),…⟩,

the dots indicating that there are infinitely more entries to come. A convenient short-form for this that we use repeatedly is:

⟨x;f⟩.

Example 12. For the iterative function f(x) = 3x(1 - x) and the seed x = 0.5, the orbit ⟨x;f⟩ is ,

⟨ 0.5, 0.75, 0.5625, 0.7383, … ⟩.

GCD The GCD, or *greatest common divisor*, of two given integers is defined to be the largest integer that divides the two integers,

Example 13. gcd(10,24) = 2, and gcd(28,42) = 14.

Algorithm An action or series of actions that is performed on an input with a specific aim in mind (i.e., with the intention of producing a desired output) is called an *algorithm*.

Example 14. The long-division square-root algorithm and the Euclidean GCD algorithm (to be discussed in Chapter 2) are both examples of

algorithms.

Symbol for Absolute Value The absolute value of a number x is written as |x| and is defined as follows: |x| = x if x ≥ 0, and |x| = -x if x < 0.

Example 15. |- 3| = |3| = 3, and |- 7| = |7| = 7.

Limit of a Sequence A sequence of numbers

$$\langle a1,a2,a3,...,an,\ldots\rangle$$

is said to have a *limit* L if, given any positive number h, however small, from some point onwards all members of the sequence lie between L - h and L + h. That is, from some point onwards, all the members of the sequence lie at a distance of no more than from L. When this happens, the sequence is said to *converge* to L.

Example 16. The sequence of reciprocals of the positive integers,

$$1, 1\ 2, 1\ 3, 1\ 4, 1\ 5,\ldots$$

converges to 0, and the sequence composed entirely of 1's,

$$1, 1, 1, 1, 1, 1, ...,$$

converges to 1.

Some examples of limits of iteration sequences are given below.

Example 17. For the function f(x) = x2 and the seed value x = 0.2, the iteration sequence ⟨x;f⟩ is

$$\langle\ 0.2,\quad 0.04,\quad 0.0016,\quad 0.00000256,\ \ldots\ \rangle,$$

and the sequence visibly converges to 0. Note however that the sequence never quite reaches the limiting value.

Example 18. For the function f(x) = 1 + x⁄2 and the seed value x = 1, the iteration sequence is

$$1,3\ 2,7\ 4,15\ 8\ ,31\ 16,\ 63\ 32,127\ 64\ ,\ldots,.$$

and the sequence converges to 2. As in the earlier example, no member of the sequence actually equals the limit. Note that this sequence can be written in the more instructive form

$$1,\ 1 + 1\ 2,1 + 1\ 2 + 1\ 4,1 + 1\ 2 + 1\ 4 + 1\ 8,1 + 1\ 2 + 1\ 4 + 1\ 8 + 1\ 16,\ldots.$$

Example 19. For the function f(x) = 1 - x and the seed value x = 0, the iteration sequence is

$$\langle 0,1,0,1,0,1,0,1,0,1,\ldots\rangle.$$

The sequence has no limit in this instance.

Example 20. For the function f(x) = x3 and the seed value x = 2, the interaction sequence is

$$\langle 2,8,512,134217728,\ldots\rangle.$$

The sequence clearly increases without limit—the next term is equal to

$$241785163922925834941235 2;$$

whew ! This is sometimes expressed by writing: *the limit of the sequence is infinity*.

Remark. The idea of a limit is a subtle one, and you may find it somewhat confusing in the beginning. We will meet the concept quite often in the pages to follow, though informally. Much later, when you are studying calculus, you will find that a proper, *formal* understanding of the concept of limit is indispensable.

Rem (a÷b). Let a,b be integers, with b non-zero. The remainder left when a is divided by b is denoted by Rem (a ÷ b)(Rem' for 'remainder'). This is a very convenient symbol that we shall use quite often.

Example 21. Rem (20 ÷ 7) = 6, Rem (12 ÷ 3) = 0.

Fixed Point Let *f* be an iteration function. A number a such that f(a) = a is called a *fixed point* of f, because f "fixes" a. The orbit ⟨a;f⟩ in this case is

$$\langle a,a,a,a,a,a,\ldots\rangle.$$

Example 22. If the function is $f(x) = x^2 - 2$, then 2 is a fixed point, because f(2) = 22 - 2 = 2.

Attracting Fixes Point, Attractor A fixed point α for a given iteration function f is described as an attractor if for all seeds x0 sufficiently close to α, the sequence ⟨x0;f⟩ converges to α. The fixed point may be said.to exert a

strong 'gravitational pull' on all nearby points.

Example 23. Let f(x) = 6 -1 2x; then 4 is an attracting fixed point. Thus, starting at the seed value 0, we get the orbit

$$\langle 0,6,3,4.5,3.75,4.125,3.9375,4.03125,3.984375\ ,\ \ldots\ \rangle.$$

The orbit converges to 4.

Repelling Fixed Point, Repeller A fixed point α for a given iteration function f is described as a repeller if for all seeds x0 sufficiently close to α but not equal to α, the sequence ⟨x0;f⟩ fails to converge to α. (Of course, if x0 = α then there is no problem—the sequence of iterates simply stays fixed at the same point, α.)

Example 24. Let f(x) = 6 - 2x; then 2 is a repelling fixed point. Thus, starting at the seed value 0, we get the orbit

$$\langle 0,6,\ -\ 6,18,\ -\ 30,66,\ -\ 126,258,\ -\ 510,1026,\ldots\rangle.$$

The orbit diverges.

Remark. Later (in Chapters 15 and 16, and again in Volume II) we shall find it necessary to distinguish between attractors and *weak attractors*, and between repellers and *weak repellers.*

Cycle An iteration sequence that returns to its starting value after a certain number of steps is called a *cycle*. The number of entries in the cycle is the *length* of the cycle.

Example 25. For the function f(x) = 1 - x, the orbit of 2 is

$$\langle 2,\ -\ 1,2,\ -\ 1,2,\ -\ 1,2,\ -\ 1,\ldots\rangle.$$

The "string" ⟨2,-1⟩ is a cycle with a length of 2.

Example 26. Let the iterative function be the following:

> *Add* 1 *to the input, divide the sum by* 3, *and let the output be the remainder thus obtained.*

In symbols: if the input is x, then the output is given by f(x) = Rem ((x + 1) ÷ 3). Choose an input, say x = 2; the orbit ⟨2; f ⟩ is then

$$\langle 2,0,1,2,0,1,2,\ldots\rangle.$$

The string (0,1,2⟩ is a cycle with a length of 3.

2-cycle A cycle with a length of 2 is called a 2-cycle. (The first example presented above shows a 2-cycle.) A 2-cycle therefore has the form

$$\langle a,b\rangle,$$

with a≠b.

n-cycle A cycle with a length of n, where n is some positive integer, is called a

n-cycle. A fixed point can be thought of as a 1-cycle. A 3-cycle has the form

$$\langle a,b,c\rangle,$$

where a,b and c are unequal.

SSQ(*n*) The sum of the squares of the digits of the integer n is denoted by the symbol SSQ (*n*). (All computations are with reference to base-10.)

Example 27. SSQ(23) = 22 + 32 = 13. The iteration using this function produces some pretty results, as we shall see in a later chapter.

SUM(*n*) The sum of the digits of the integer n is denoted by the symbol SUM (*n*).

Example 28. SUM(125) =8, SUM(139) =13.

Composition of Functions Let *f* and *g* such that the quantity *f*(*g*(*x*)) can be meaningfully computed for each input x (note that g acts first, and then f acts on the output given by g):

$$x \longrightarrow \boxed{g} \longrightarrow g(x) \longrightarrow \boxed{f} \longrightarrow f(g(x)).$$

Then a new function can be defined in which the output f(g(x)) is produced "in a single step". This function is denoted by the symbol f ◦ g:

$$x \longrightarrow \boxed{f \circ g} \longrightarrow f(g(x)).$$

We say that f ◦ g has been produced by *composing* the functions f and g (in the given order: g first, then f).

Example 29. Let f(x) = x2,g(x) = x + 1, the domains of both f and g being

the set of positive integers **N**. Then

$$f \circ g(x) = (x + 1)2, g \circ f(x) = x2 + 1.$$

Thus, $f \circ g(10) = f(11) = 121$, and $g \circ f(10) = g(100) = 101$. Note that $f \circ g(10)$ is not the same *a*s $g \circ f(10)$.

In general, for arbitrary functions f and g, it will be the case that $f \circ g$ and $g \circ f$ are distinct functions. In some instances, however, it happens that $f \circ g = g \circ f$; that is, $f \circ g(x) = g \circ f(x)$ for every possible value of x. In this case we say that *f and g commute.*

Example 30. The functions $f(x) = x2$ and $g(x) = x3$ commute, because for any x,

$$f \circ g(x) = (x3)2 = x6, g \circ f(x) = (x2)3 = x6.$$

More generally, any two power functions commute.

Remark. We can compose functions in arbitrarily complicated ways. For example, for the functions $f(x) = x2$ and $g(x) = x + 1$, we have:

$$f \circ g \circ f(x) = (x2 + 1)2,$$

$$g \circ g \circ g(x) = x + 3,$$

$$g \circ g \circ f \circ g(x) = (x + 1)2 + 2,$$

and so on.

Square Root of a Function Let two functions a and b be related in such a way that $b \circ b = a$, that is, $b(b(x)) = a(x)$ for all values of x. Then b is called a *square root* of a.

Example 31. For the function $a(x) = x + 2$, the function $b(x) = x + 1$ is a square root, because for any x,

$$b(b(x)) = b(x + 1) = (x + 1) + 1 = x + 2 = a(x).$$

Some functions do not have square roots at all (we could call them "rootless"), and some have more than one square root.

Binomial Theorem for positive Integral Indices This states that if n is a positive integer, then the expansion of $(1 + x)^n$ is given by

$$1 + nx + n(n - 1)\,2!\,x2 + n(n - 1)(n - 2)\,3!\,x3 + n(n - 1)(n - 2)(n - 3)\,4!\,x4 + \cdots,$$

where 2! = 2,3! = 6,4! = 24, and in general, n! (pronounced n factorial") is given by

$$n! = 1 \times 2 \times 3 \times \cdots \times (n - 1) \times n.$$

Note that (1 + x)n is a polynomial in x of degree n. (The sequence of factors n,n - 1,n - 2,…, reaches zero at some stage—because n is a positive integer—and the expression then terminates.)

Example 32. We have the following binomial expansions:

$$(1 + x)2 = 1 + 2x + x2,$$

$$(1 + x)3 = 1 + 3x + 3x2 + x3,$$

$$(1 + x)4 = 1 + 4x + 6x2 + 4x3 + x4,$$

$$(1 + x)5 = 1 + 5x + 10x2 + 10x3 + 5x4 + x5.$$

Binomial Theorem for Arbitrary Indices This states that if n is any real number, and $-1 < x < 1$,then (1 + x)nmay be expanded in powers of x as follows:

$$1 + nx + n(n - 1)\,2!\,x2 + n(n - 1)(n - 2)\,3!\,x3 + n(n - 1)(n - 2)(n - 3)\,4!\,x4 + \cdots.$$

If n is not a positive integer or 0, the sequence of terms goes on indefinitely, and the series is now an *infinite series*. The condition $-1 < x < 1$ cannot be dispensed with.

Example 33. With n = 1⁄2 and n = 1⁄3, we get infinite series for (1 + x)1⁄2 and (1 + x)1⁄3, respectively, valid for –l < x < 1:

$$(1 + x)1/2 = 1 + x\,2 - x2\,8 + x2\,16 - 5x4\,128 + 7x5\,256 - \cdots,$$

$$(1 + x)1/3 = 1 + x\,3 - x2\,9 + 5x3\,81 - 10x4\,243 + 22x5\,729 - \cdots.$$

Lattice Point A point in the coordinate plane both of whose coordinates are integers is known as a *lattice point*.

Example 34. The points with coordinates (2, 3), (5, –1) and (10,0) are lattice points, whereas (2, 0.5) is not a lattice point.

Monic Polynomial A one-variable polynomial in which the leading term

(i.e., the term with highest degree) has a coefficient of 1 is known *as* a *monic polynomial*.

Example 35. The polynomials f,g given by

$$f(x) = x3 - 3x + 1, g(x) = x100 + 5x37 + x + 1,$$

are monic, whereas 2x3- x2 + x + 1 is not monic. Note that the product of two monic polynomials in the same variable is monic.

★ ★

In Chapter 13 we shall study a variety of iterations set in a purely geometric context. Various geometric terms will be met in that chapter, e.g.,

- *median* and *centroid* of a triangle;
- *incenter* and *incircle* of a triangle;
- *altitude* and *orthocenter* of a triangle;
- *pedal triangle* of a point within a triangle;
- *oriented area* of a polygon;
- *triangle of reference*, and *barycentric coordinates*.

As these terms will not be encountered outside Chapter 13, we defer the definitions to that chapter itself.

Chapter 3

Examples of Iterations

3.1 The GCD algorithm

A nice example of an iterative process at work is seen in the division algorithm to extract the gcd ('greatest common divisor') of two given integers. (Recall that the gcd of two integers a,b, denoted by the symbol gcd(a,b), is defined to k the largest integer that divides both a and b. For example, gcd(92,115) = 23 and gcd(111,185) = 37.)

The gcd algorithm goes back to Greek times, and a description can be found in Euclid's classic text, THE ELEMENTS; indeed, the algorithm is still referred to as the **EUCLIDEAN ALGORITHM**. Here is a worked example to remind you how it works.

Example 1. To find the gcd of 25 and 165, the calculations are set out as shown below.

$$165 = (6 \times 25) + 15,$$

$$25 = (1 \times 15) + 10,$$

$$15 = (1 \times 10) + 5,$$

$$10 = (2 \times 5) \text{ (no remainder left).}$$

Therefore, gcd(25,165) = 5. Do you follow what exactly is "happening"?

How the GCD algorithm works

What "happens" is this the algorithm accepts two non-negative integers as INPUT and then replaces them, using an iterative rule (which we shall

describe shortly), by two other integers, referred to as the OUTPUT. This is done in such a manner that the output pair is smaller than the input pair, yet has the same gcd. More precisely: the larger integer in the output pair is strictly smaller than the larger integer in the input pair. This procedure is repeated several times. Eventually we obtain a number pair in which one of the numbers is zero and the other number is the gcd.

Here is a description of the iterative rule

- Write the input pair as (a,b), where a and b are non-negative integers and $a \geq b$..
- Divide a by b, and denote the remainder in the division by c.
- Declare the pair (b,c) to be the output.

Example 2. The input (100, 80) produces the output (80, 20); the input (100, 16) produces the output (16, 4); and so on.

Now repeat the process with the new pair as input. Continue this till at some stage the remainder is zero. In other words, repeatedly replace the pair

(larger number, smaller number)

by the pair

(smaller number, remainder),

until at some point the division is exact.

Example 3. To illustrate the iteration, we examine its action on the input pair (289, 34). Dividing 289 by 34, we obtain a quotient of 8 and a remainder of 17:

$$289 = (34 \times 8) + 17.$$

The new pair is therefore (34, 17). Next, dividing 34 by 17, we obtain quotient of 2 and a remainder of 0:

$$34 = (2 \times 17) + 0,$$

so the next pair is (17, 0). As the zero-remainder stage has now been reached, we declare the gcd of the two numbers to be 17. As a check, note that 289 = 17

× 17 and 34 = 2 × 17, and 17 and 2 are coprime. (Integers that share no factor greater than 1 are said to be *coprime*.)

A pictorial manner of representing the basic operation involved is given below:

$$(a,b) \rightarrow \boxed{\text{GCD MACHINE}} \rightarrow (b, \text{Rem}(a \div b)),$$

where the symbol 'Rem (a ÷ b)' refers to the remainder left in the division a ÷ b.

Example 4. Rem (100 ÷ 19) = 5, Rem (1000 ÷ 17) = 14.

Computer implementation of the GCD algorithm

For those familiar with "pseudo-code", in which computer programs are often written, the gcd algorithm is presented as follows:

Step 0: Read the positive integers a and b.

Step 1: Compute c = Rem (a ÷ b).

Step 2: If c = 0 then print the value of b and stop. The printed number is the gcd.

Step 3: If c > 0 then replace (a,b) by (b,c).

Step 4: Go back to Step 1.

The instruction in Step 4 (Go Back to Step 1') is the iterative part of the algorithm; it recurs *a*s long as c≠0.

Here is a computer program, written in ***BASIC***, that performs the actions described above.

```
REM a, b are positive integers
INPUT a,b
c = a - b* INT (a/b)
WHILE c > 0
    a = b : b = c : c = a - b* INT(a/b)
```

```
    WEND
    PRINT "GCD of " ; a ; " and " ; b ; " = " ; b
```

To illustrate how the algorithm works, we study its effect on the two input pairs (17, 10) and (82, 64), The results are shown below, with the values of a,b and c listed on the right at each stage.

Example 5. (a,b) = (17,10)

Stage	a	b	$c = \text{Rem}(a \div b)$
1	17	10	7
2	10	7	3
3	7	3	1
4	3	1	0
5	1	0	—

Therefore, gcd(17,10) = 1.

Example 6. (a,b) = (82,64)

Stage	a	b	$c = \text{Rem}(a \div b)$
1	82	64	18
2	64	18	10
3	18	10	8
4	10	8	2
5	8	2	0
6	2	0	—

Therefore, gcd(82,64) = 2.

Exercises

3.1.1 At the conclusion of Step 2, is it always true that c < b?

3.1.2 Apply the gcd algorithm to the following number pairs: (a) (100, 63) (b) (100, 78) (c) (100, 17).

3.2 Squares

In this section, we study the following iteration:

Replace the input number by its square.

So if the input is x, then the output is x2. An initial input of 2 gives rise to the numbers shown below:

$$\begin{aligned} 2^2 &= 4, \\ 4^2 &= 16, \\ 16^2 &= 256, \\ 256^2 &= 65536, \\ 65536^2 &= 4294967296, \\ 4294967296^2 &= 18446744073709551616, \end{aligned}$$

and so on. Visually:

$$2 \mapsto 4 \mapsto 16 \mapsto 256 \mapsto 65536 \mapsto 4294967296 \mapsto \cdots.$$

Note how rapidly the numbers increase. This will obviously happen whenever the starting number exceeds 1: the resulting sequence "explodes"—that is, the numbers in the sequence increase with great rapidity and without limit. If, however, the starting number lies between–1 and 1, then the resulting sequence races with great speed towards a rather obvious limit. For example, a starting value of 0.1 gives rise to the following sequence:

$$\langle 0.1, 0.01, 0.0001, 0.00000001, 0.0000000000000001, \ldots \rangle,$$

and it is easy to guess the limit (it is clearly 0). Note, though, that the sequence never quite gets to this limit—it is always a 'little bit' greater than this number.

Exercises

3.2.1 Suppose that with an initial input of x, the process of squaring is repeated n times, where n is some positive integer. Find a formula for the final number thus obtained, in terms of x and n.

3.2.2 Show that if the initial input exceeds 1, then the numbers in the iteration sequence increase without limit, that is, *the limit of the sequence is infinite.*

(Here is one way of doing the problem. We start by stating that if h is

any positive quantity, and n is any positive integer greater than 1, then

$$(1 + h)n. > 1 + nh.$$

Example 7. If h = 0.5 and n= 10, then (1 + h)n = 1.510 ≈ 57.6650,1+ nh = 1 + 5 = 6,. and of course, 57.6650 > 6.

The inequality is not too difficult to prove. Let n = 2; then (1 + h)n = (1 + h)2 = 1 + 2h + h2. Since h2 is positive, 1 + 2h + h2 is greater than 1 + 2h. so (1 + h)2 > 1 + 2h.

We now use this result to prove the result for the next case, n = 3. Thus:

$$\begin{aligned}(1 + h)3 &= (1 + h)2 \cdot (1 + h)\\ &> (1 + 2h) \cdot (1 + h)\\ &= 1 + 3h + 2h2\\ &> 1 + 3h.\end{aligned}$$

The results (1 + h)4 > 1 + 4h,(1 + h)5 > 1 + 5h,…, follow similarly. Though this does not show that (1 + h)n > 1 + nh for all positive integers n, the reader will readily see how the proof can be written down.

Now consider our iteration sequence, in which each number after the first is the square of the previous one ($x \mapsto x2$). After n repetitions of the squaring process, we obtain the number x2n

Write h for x - 1; then h is positive, since x > 1. Now write x2n a*s* (1 + h)2n , and apply the inequality stated above. Here is what we obtain:

$$x2n = (1 + h)2n > 1 + 2nh.$$

Since 2nh doubles each time n increases by 1, it follows that 2nh increases without limit, so the quantity x2n too increases without limit. Therefore the limit of the sequence is infinite.)

3.2.3 Show that if the initial input value lies between–l and 1, then the limit of the iteration sequence is 0. (Hint: Use the result of the previous exercise.)

3.2.4 With an initial input of 10, how many repetitions of this process would

be needed in order to produce an integer that has more than a thousand digits? Ten thousand digits?

3.2.5 With an initial input of 2, how many repetitions of this process would be needed in order to produce an integer that has more than a thousand digits? Ten thousand digits?

3.2.6 What happens if in this iteration we replace the word 'square' by the word 'cube'?

3.3 Square roots

In the last section a calculator would have proved useful. In the next iteration, access to a calculator or computer is essential.

We start with any positive number as input. The iterative rule is:

Replace the number by its square root.

So if the input is x, then the output is x; we write $x \mapsto x$. On a calculator, we enter the number x into the display and then repeatedly press the SQUARE ROOT key (the one marked). (On some calculators the '=' or 'EXE' key must be pressed each time for the computation to proceed.) Here is the result of the first few rounds of the iteration when we start with an initial input of 10:

3.1623...,	1.7783...,	1.3335...,
1.1548...,	1.0746...,	1.0366...,
1.0182...,	1.009...,	1.0045...,
1.0023...,	1.0011...,	1.0006....

(Read the numbers row-wise: starting with 10, we obtain 3.1623, followed by 1.7783, 1.3335, 1.1548, 1.0746 and so on.)

Note that the sequence seems to be heading towards a definite destination. Alas, however, it will never quite get to this destination. (Do you see why?)

Exercises

3.3.1 What is the limit referred to above? Will the limit remain the same if we change the starting number? Experiment with different starting

numbers before stating any conclusion.

3.3.2 What happens if we replace the word ‘square root’ by ‘cube root’?

3.3.3 What is the relation between this iteration and the previous one?

★★★★★★★★★★★★★★★★★★★

In the pages to follow, you will see many examples of iterations. Each iteration gives rise to a certain sequence, and the task before you is to investigate the patterns hidden within the sequence. Have fun!

Chapter 4

The Four-Numbers Game

4.1 The iteration

Choose any four integers and arrange them in any order. For example, we may choose 41, 13, 92 and 7 and arrange them in the order given. An ordered foursome of numbers is referred to by mathematicians a **QUADRUPLE**, and it is usually enclosed in parenthesis thus: (41, 13, 92, 7). We use this quadruple as the seed of the iteration, replacing it by another quadruple according to the following rule:

$$(a,b,c,d)\mapsto(|a - b|,|b - c|,|c - d|,|d - a|).$$

(The double-bar symbol ' ||' stands for **ABSOLUTE VALUE**: $|- 3| = 3$,$|- 19| = 19$, $|3 - 7| = 4$, and so on. Note the cyclic order in which differences are taken.)

Example 1. The quadruple (41, 13, 92, 7) yields (28, 79, 85, 34), because

$$|41 - 13| = 28,|13 - 92| = 79,$$

$$|92 - 7| = 85,|7 - 41| = 34.$$

Since we take only absolute differences, the output can never contain a negative number.

We now iterate the operation. Here is the result:

Stage	Quadruple
1	(41, 13, 92, 7)
2	(28, 19, 85, 34)

3	(51, 6, 51, 6)
4	(45, 45, 45, 45)
5	(0, 0, 0, 0)

Note that once we reach the quadruple (0,0,0,0), we remain there; we are "stuck"!

Let us do the iteration with some other seed, say (19,27,63,90). Here is the result:

Stage	Quadruple
1	(19,27,63,90)
2	(8,36,27,71)
3	(28, 9, 44, 63)
4	(19,35,19, 35)
5	(16, 16, 16, 16)
6	(0, 0, 0, 0)

We are stuck once again. Surely something curious is happening?

4.2 Exercises

4.2.1 Try out the procedure on each of the following seeds:

(i) (1, 17, 63, 6)

(ii) (91, 117, 13, 60)

(iii) (236,40, 67, 3)

(iv) (47, 3, 102, 200)

What do you notice? Do you end up with the same result in each case (namely, (0,0, 0,0))?

Remark. If this is indeed so, we have found a type of problem where it does not matter if you make a mistake halfway through—you still end up with the right answer!

4.2.2 Identify the quadruples that reach (0,0,0,0) after (i) one round (ii) two rounds (iii) three rounds of the iteration.

4.2.3 Here is a simple computer program in *BASIC* that will help you to do some of these exercises on the computer.

```
REM Four numbers game
INPUT a,b,c,d
x = a : y = b : z = c : w = d
count = 0
WHILE x + y + z + w > 0
    xl = ABS (x -y) : yl = ABS (y -z)
    zl = ABS (z-w) : wl = ABS (w -x)
    count = count + 1
    x = xl: y = yl: z = zl: w = wl
    WEND
    PRINT "( 0, 0, 0, 0 ) reached in " ; count ; " steps"
```

What role is played by the statement "while x + y + z + w > 0?"

4.2.4 What if we work with **TRIPLES** rather than with quadruples?

Example 2. Starting with the triple (3, 17, 40), we get:

Stage	Triple
1	(3, 17, 40)

2	(14, 23, 37)
3	(6, 14, 23)
4	(8, 9, 17)
5	(1, 8, 9)
6	(7, 1, 8)

and so on. In symbols:

$$(a,b,c) \mapsto |a - b|,|b - c|,|c - a|.$$

Try out the iteration on the following seeds:

(i) (2, 13, 17);

(ii) (3, 19, 50);

(iii) (100, 201, 18).

What do you notice? Does each triple ultimately reach (0,0,0)?

4.2.5 Make a list of the triples that:

(i) ultimately reach (0,0,0);

(ii) ultimately reach (1,0,1) and then keep 'cycling' round and round;

(iii) cycle round some other terminal triple.

What patterns do you find? (Caution: You will need to experiment with many triple before drawing any conclusions.)

4.2.6 What happens in the case of 5-tuples and 6-tuples? Experiment around on your own—you may discover a few surprising facts!

4.3 Concluding remarks

It is important to realize that at this stage, we cannot be certain that no matter which quadruple we start with, we shall inevitably reach (0, 0, 0, 0) after some

iterations; it is only a **HYPOTHESIS** (that is, a surmise, or a guess). It is important to keep this point in mind. You may wish to work on this question: *How can one know the truth* "*for sure*" ? We shall be taking up such questions somewhat later, in Chapter 14.

Chapter 5

Sums of Squares

5.1 The iteration

Given an integer n (the input), we produce another integer (the output) by adding the squares of the digits of n. (Throughout, we work only in base-10, that is, the "decimal system.") Let SSQ (n) denote this sum (for example, SSQ(15) = 26 and SSQ(32) = 13) ; then the iteration is as shown below:

$$n \mapsto SSQ(n).$$

Example 1. $61 \mapsto 62 + 12 = 37, 37 \mapsto 32 + 72 = 58$, and so on. If we tabulate the orbits of different seeds, we soon notice some curious results.

If we start with 0, we get stuck there (02 = 0); we only produce more 0's. Likewise, if we start with 1, we produce only 1′s.

More interesting results are obtained when we start with 2: we get 4, followed by 16, then 37 and after that:

$$58, 89, 145, 42, 20, 4;$$

we have now returned to the number 4! Obviously from this point on we shall remain in cycle:

$$4 \mapsto 16 \mapsto 37 \mapsto 58 \mapsto 89 \mapsto 145 \mapsto 42 \mapsto 20 \mapsto 4,$$

and we are back at the starting point. Do you see why we use the term CYCLE?

5.2 Exercises

5.2.1 Start with each of the following seeds and see if you reach any cycle: (i) 5 (ii) 6 (iii) 8 (iv) 11 (v) 12.

5.2.2 You will probably have discovered something surprising: the cycle listed above,

$$\langle 4,16,37,58,89,145,42,20\rangle,$$

seems to occur for just about any seed! The only other cycles seem to be $\langle 1\rangle$ and $\langle 0\rangle$, though there is not much to be said about $\langle 0\rangle$; it can arise only from a seed of 0.

Identify a*s* many seeds as you can that ultimately reach the cycle $\langle 1\rangle$; there are quite a few of these seeds! See if you can capture all such seeds that are less than 100.

5.2.3 [*] Challenge!—Find procedure for generating all seeds that eventually reach the cycle $\langle 1\rangle$.

5.2.4 What if we work with cubes instead of squares?—that is, we add the *cubes* of the digits rather than the squares:

$$n\mapsto \text{ sum of the cubes of the digits of } n.$$

Example 2. $21\mapsto 23 + 13 = 9{,}9\mapsto 93 = 729$. The numbers to follow are: $73 + 23 + 93 = 1080$,$13 + 03 + 83 + 03 = 513$,$53 + 13 + 33 = 153$, $13 + 53 + 33 = 153$,…. Thus:

$$21\mapsto 9\mapsto 729\mapsto 1080\mapsto 513\mapsto 153\mapsto 153\mapsto 153\mapsto\cdots.$$

What a pleasant surprise!—we have found a number that equals the sum of the cubes of its digits:

$$153 = 13 + 53 + 33.$$

Try the iteration out on different seeds and list down the cycles you find. Find as many numbers as possible that equal the sums of the cubes of their digits. (There are only a few numbers that share this property with 153.)

5.2.5 Let us try the experiment with *fourth powers* instead of cubes. Now we get: $21\mapsto 24 + 14 = 17{,}17\mapsto 14 + 74 = 2402$, etc., and the numbers to follow are $16 + 256 + 0 + 16 = 288$, then $16 + 4096 + 4096 = 8208$; following this we get $4096 + 16 + 0 + 4096 = 8208$, and the number stays fixed at 8208 thereafter. We have found a number that equals the

sum of the fourth powers of its digits !

$$8208 = 84 + 24 + 04 + 84.$$

Try to find further examples of this property.

5.2.6 [*] Challenge!—Show that there do not exist any numbers (other than 0 and 1) that are equal to the sums of the squares of their digits.

5.3 Further comments

The following comment echoes the one made at the end of the last chapter: for the SSQ iteration too, we do not at this point know *for sure* that every seed results in one of the cycles $\langle 0 \rangle$,$\langle 1 \rangle$ and $\langle 4,16,\ldots,20 \rangle$. Not knowing "for sure" means that at this stage our knowledge is incomplete and not quite certain; it is only a hypothesis to say that we shall inevitably end up in one of the three cycles. It is true that the hypothesis has been verified in every case examined till now, but there is always a possibility of its failing in some exceptional instance. (Mathematicians refer to such instances in general as "counterexamples".) To make sure that no counterexamples exist, a very different type of analysis needs to be done. Those who are keen on see how this more challenging problem is tackled will need to wait till Chapter 14.

Chapter 6

Fixed Points and Cycles

6.1 Fixed points

Earlier we found that for the sum-of-cubes iteration,

$$153 \mapsto 153;$$

for the sum-of-fourth-powers iteration,

$$8208 \mapsto 8208;$$

and for the four numbers iteration,

$$(0,0,0,0) \mapsto (0,0,0,0).$$

Observe that in each case, *the output is identical to the input*. When this happens, the input is referred to as a **FIXED POINT** for the iteration under study. A formal definition of the term "fixed point" is given below.

Fixed Point A fixed point for the iteration $x \mapsto f(x)$ is any solution of the equation $f(x) = x$. (If a is any solution of this equation, then f "fixes" a, hence the choice of the word.

Observe that we said *any* fixed point. It is possible for a function to have many fixed points.

Example 1. For the function f(x) = x2, the fixed points are 0 and 1, because the solution set of the equation x2 = x is {0,1}.

Example 2. The function f(x) = x + 1 has no fixed point.

Example 3. The functions f(x) = x2 + 1 and g(x) = x4 + 1 have no fixed points. (This is true only because we confine our attention to the REAL

NUMBERS. If we enlarge our canvas to include COMPLEX NUMBERS, then f and g do have fixed points.)

Example 4. The SSQ function (n↦ sum of the squares of the digits of n) has just one fixed point, namely 0. (Naturally, this statement needs proof.)

1-cycle If a is a fixed point for the function f, then ⟨a⟩ is a 1-cycle for the iteration x↦f(x).

Exercises

6.1.1 Consider the function f that takes x to x2 – 2; thus 10 ↦98,7↦47, and so on. What are the fixed points of f?

6.1.2 What are the fixed points of the function f given by f(x) = 2x –x2?

6.1.3 What are the fixed points of the following functions f?

(a) f(x) = x2 – 12 (b) f(x) = x3 – 3x?

6.1.4 Prove that the function f(x) = x2 + 1 has no (real) fixed points.

Hint. Show that x2 + 1 > x for all real values of x.

6.1.5 Prove that the function g(x) = x4 + 1 has no (real) fixed points.

6.2 Cycles

Just as we have l-cycles, we also have 2-*cycles, 3-cycles*, and so on.

- *2-cycles*: if a and f(a) are unequal, but the second iterate yields the original number, i.e., f(f(a)) = a, then we get a 2-cycle:

$$\langle a,f(a)\rangle;$$

- *3-cycles*: if a,f(a) and f(f(a)) are unequal, but the third iterate yields the original number, i.e., f(f(f(a))) = a, then we get a 3-cycle:

$$\langle a,f(a),f(f(a))\rangle;$$

- *4-cycles*: if a,f(a),f(f(a)) and f(f(f(a))) are all unequal,but the fourth iterate yields the original number, i.e., f(f(f(f(a)))) = a, then we get a 4-cycle:

$$\langle a,f(a),f(f(a)),f(f(f(a)))\rangle;$$

and so on. The sketches below convey these ideas in a visual and more appealing language.

1-cycle: $a \mapsto a$;

2-cycle: $a \mapsto f(a) \mapsto a$;

3-cycle: $a \mapsto f(a) \mapsto f(f(a)) \mapsto a$;

4-cycle: $a \mapsto f(a) \mapsto f(f(a)) \mapsto f(f(f(a))) \mapsto a$.

Repeating decimals

Repeating cycles have surely been seen earlier by the reader, for they often arise in simple situations. For instance, consider the phenomenon of recurring decimals. Displayed below are some familiar instances of recurring decimals.

$1/3 = 0.333333...$

$1/7 = 0.\ 142857142857...$

$5/12 = 0.416666...$

$1/13 = 0.076923076923...$

The repeating portion for 1/7 is the 'string' 142857; note that its length is 6. Where is the 'cycle' in this example? To find it, examine the division $1.0000 \ldots \div 7$. You will find that the remainders go through the cycle

$$\langle 3, 2, 6, 4, 5, 1 \rangle;$$

that is, 1 is followed by 3, then 3 is followed by 2, and so on.

Exercises

6.2.1 Find by computation the lengths of the repeating portions of the decimals corresponding to the fractions (i) 1/19 (ii) 1/21 (iii) 1/29.

6.2.2 Find the fraction a/b whose decimal expansion is

$$a\ b = 0.415757575757\ldots.$$

(The repeating portion is the string 57.)

6.2.3 Find a fraction a/b whose repeating portion has a length of (i) 5 (ii) 16.

Cycles in the sequence of units digits

Another example of a cycle is found in the sequence of units digits of the powers of 2. To obtain the sequence, we first list the successive powers of *2*:

2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, ...

then retain only the units digits of these numbers:

2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, ...

We note that the cycle ⟨2, 4, 8, 6⟩repeats indefinitely.

Likewise, if we list the successive powers of 3:

3, 9, 27, 81, 243, 729, *2*187, 6561, 19683, ...,

then retain only their units digits:

3, 9, 7, 1, 3, 9, 7, 1, 3, ...,

we find that the cycle ⟨ 3, 9, 7, 1⟩ recurs indefinitely. It is an easy matter to construct a large number of such examples.

Exercises

6.2.4 Find the cycles in the units digits of (i) the powers of 7 (ii) the powers of 8.

6.2.5 Explain why the cycle⟨ 2,4, 8, 6 ⟩ occurs in the sequence of units digits of the powers of 2.

6.2.6 Explain why the cycle ⟨3,9,7,1⟩ occurs in the sequence of units digits of the powers of 3.

6.2.7 (Challenge!) Explain the phenomenon of recurring decimals.

More precisely, explain why for any two positive integers *a,b* the decimal obtained in the division a ÷ b will necessarily repeat exactly from some point on. (If the division is exact, then the repeating portion will be a single 0, that is, the decimal will have the appearance...000.... For example, if a = 43, b = 25, then a ÷ b = 1.72000….)

Chapter 7

The Kaprekar Constant

The mathematics of great amateurs The iteration we study in this chapter is associated with the name of a famous amateur mathematician (by profession he was a teacher in a high school) from the state of Maharashtra in India, Shri D. R. Kaprekar. What does "amateur mathematician" mean? Perhaps it refers to a person who loves doing mathematics for its own sake. Kaprekar loved playing with numbers, and he loved conducting investigations and uncovering unusual properties of numbers. Thus he came to discover what we now refer to as the 'Kaprekar iteration' and the 'Kaprekar constant'. Indeed, he discovered many phenomena of the type that we shall describe below.

Remark. It is perhaps not so very difficult to discover things the way Kaprekar did;what is required, more than anything else, is a deep love of one's subject, and qualities such as *perseverance* and *tenacity*: the ability to stay with an investigation once one has started it, without giving up halfway through. Perhaps this quality—that of tenacity—is one that any truly great person possesses. Readers who are familiar with the work of the great astronomer Johannes Kepler, and know of the stupendous labour that went into the discovery of the laws of planetary motion (Kepler's laws), would surely agree with this observation.

7.1 The Kaprekar iteration

Here is how the Kaprekar iteration works: starting with any whole-number seed x(initially we shall work with 4-digit numbers, written in base-10),we rearrange its digits so as to make the *largest* possible number, xmax. Next, we

rearrange the digits of x so as to make the *smallest* possible number, xmin. Clearly, the digits of xmin will be in reverse order as compared with xmax.

Example 1. If x = 2810, then xmax = 8210 and xmin = 0128. (This is the same as 128, but for reasons that will be clear shortly, we prefer to write it as 0128.)

The Kaprekar function is now defined by the following rule:

Replace x by $x_{\max} - x_{\max}$

Write *K* for the Kaprekar function; then $K\, x_{\max} - x_{\max}$. In words,

> K(x) *equals the difference between the largest number that can be formed by rearranging the digits of* x *and the smallest such number.*

The Kaprekar iteration consists of iterating this operation. To avoid ambiguities in the application of the function, we choose to impose the following rule with regard to the output:

> *If the output has fewer digits than the input, then we pad it from the front* (*the* "*left side*") *with a suitable number of zeroes so that the input and the output have the same number of digits.* (*Note that this does not alter its value as a number.*)

In other words, we restore its number of digits. With this rule, every number in the chain will have the same number of digits.

Example 2. Let x = 1039; then xmax = 9310,xmin = 0139(= 139), and therefore, K(1039) = xmax - xmin = 9310 - 0139 = 9171. Thus,

$$1039 \mapsto 9310 - 0139 = 9171.$$

Applying the Kaprekar operation to 9171, we get

$$K(9171) = 9711 - 1179 = 8532.$$

Next, K(8532) = 8532 - 2358 = 6174, and so on. The net result of the operations can be written compactly as

$$K(K(K(1039))) = 6174,$$

or, more appealingly, as:

$$1039 \to 9171 \mapsto 8532 \mapsto 6174 \mapsto \cdots.$$

Why "pad" the output with zeroes? You may wonder about the need to pad the output number with zeroes on the left side so as to make the output and input have the same number of digits. The point is that without such a rule,situations could arise where the application of the K function becomes unclear.

Example 3. For the input value x = 1112, we obtain

$$K(1112) = 2111 - 1112 = 999,$$

which is a 3-digit number, whereas 1112 has four digits. The question now is, what is $K(K(\ 1112))$? If we apply the Kaprekar operation directly to 999, we obtain 999 - 999 = 0, whereas if we apply the rule and write K(1112) as 0999, we get 9990 - 0999 = 8991, which is of the same length *a*s the number 1112. So do we write K (1112) as 999 or as 0999? Our preference will be for the latter, i.e., for padding the output with 0's. However this is on purely aesthetic grounds: it seems "nicer" to preserve the number of digits in the output as the iteration progresses. Note that the other rule ("no padding") also provides us with a perfectly valid iteration.

7.2 Exercises

7.2.1 Apply the Kaprekar iteration to 1039. Do you reach a cycle or fixed point?

7.2.2 Repeat the exercise for the seeds (i) 8619 (ii) 1234 (iii) 1235 (iv) 9000 (v) 8000. What do you notice? Can you make a guess as to the general behaviour of the Kaprekar iteration?

7.2.3 What is the largest possible value of K(x) if is a 4-digit number?

7.2.4 One of the fixed points of the K iteration is the number 0000. Which seeds reach this fixed point in one step? In two steps? In three steps?

7.2.5 The other fixed point that you will have found is the number 6174. (It is a fixed point because if x = 6174, then xmax = 7641 and xmin = 1467, therefore,xmax - xmin = 7641 - 1467 = 6174 = x; so K(6174) = 6174.) Are there any fixed points other than these two (0000 and 6174)? Experiment with different seeds before drawing any conclusion.

The fixed point 6174 is known as the **KAPREKAR CONSTANT** .

7.2.6 Experiment with 3-digit seeds. with the understanding that all numbers involved in the computation are written as 3-digit numbers.

Example 4. x = 157 leads to xmax = 751 and xmin = 157, and therefore, K(x) = xmax - xmin = 751 - 157 = 594.
Try out the Kaprekar iteration on different seeds. Are there any Kaprekar constants to be found here?

7.2.7 Prove that the 3-digit Kaprekar iteration has precisely two fixed points. (You could argue as follows. Let the digits of *x* be a,b and c, where a ≥ b ≥ c.

Then

$$\text{xmax} = 100a + 10b + c$$

$$\text{xmin} = 100c + 10b + a$$

$$K(x) = (100a + 10b + c) - (100c + 10b + a)$$

$$= 99a - 99c = 99(a - c).$$

Thus K(x) is always a multiple of 99, and so a fixed point of K is necessarily a multiple of 99. Now there are just ten 3-digit multiples of 99, and it is an easy matter to check whether any of them is a fixed point of K. This procedure will obviously give us all the fixed points we seek.)

7.2.8 Try out the 5-digit case on your own; see if there are any more such constants to be found. Keep a careful record of the patterns that you discover.

7.2.9 Suppose that we did not insist on padding the front of the number with 0's so as to preserve the number of digits. What can be said about this iteration?

7.3 Concluding remarks

Projects for the student For those who have access to a computer and are able to write simple computer programs in *BASIC* or *PASCAL*, the

computational work can be lessened considerably by working on a computer. Problems that are computationally difficult but are of intrinsic interest can be taken up, for example:

- For the 4-digit Kaprekar iteration, find the largest number of steps needed to reach the Kaprekar constant 6174.
- Investigate which are the most "resilient" numbers—the ones whose orbits take the longest to reach the Kaprekar constant.
- Investigate whether there are Kaprekar constants for 6-digit numbers and for 7-digit numbers.
- What about extensions to bases other than 10? Investigate this problem too.

7.4 Recent results

In 1978 two mathematicians (H. Hasse and G. D. Pritchett) extended Kaprekar's work to bases other than base 10 and came up with the following pretty finding: A *Kaprekar constant exists for 4-digit numbers in base* B *if and only if* B *is a number of the from* $5 \times 2k$ *where* k *is* 0 *or an odd positive number, i.e., if and if B is one of the numbers*

$$5,10,40,160,640,....$$

Moreover, if B *is one of these numbers, then the Kaprekar constant may be obtained as follows. Let* $c = B/5$; *then the 4-digit Kaprekar constant is the number whose digits, reading from left to right, are* $3c,c - 1,4c - 1$ *and* $2c$, *respectively; i.e., the number*

$K = 3c,c - 1,4c - 1,2c^{-}$.

Some examples are given below.

- For B = 10, we get c = 2 and K = 6174. (This agrees with what we already know.)
- For B = 5, we get c = 1 and K = 3032. For example, the seed x = (1342)5 yields:

$$(1342)5 \mapsto (4321)5 - (1234)5 = (3032)5.$$

The constant has been reached in a single step.

For other bases, there is no single number to which all orbits converge; there are, instead, multiple fixed points (we could, if we wish, call them "Kaprekar constants") and "Kaprekar cycles". Thus:

- In base 2 there are two fixed points, namely (0111)2 and (1001)2. Every orbit converges to one of these two fixed points.
- In base 4 there is a fixed points(3021)4, but there is also a Kaprekar 2-cycle:

  $$(2022)4,(1332)4\ .$$

- In base 3 we have the Kaprekar 2-cycle

  $$(1221)3,(1012)3\ .$$

- In base 6 we even have a Kaprekar 6-cycle!—

  $$(3043)6,(3552)6,(3133)6,(1554)6,(4042)6,(4132)6\ .$$

- In base 7 we have the Kaprekar 3-cycle

  $$(3054)7,(5052)7,(5232)7\ .$$

- In base 9 we have the Kaprekar 3-cycle

  $$(5074)9,(7072)9,(7432)9\ .$$

- If the base B is divisible by 5 but has a prime factor other than 2 and 5, then the number K given by

  $$K = 3c,c - 1,4c - 1,2c^{-} \quad (\text{where } c = B/5)$$

  is a fixed point, but there are Kaprekar cycles too.

The reader is invited to find more such results, via further experimentation.

7.5 Biographical details

It may be appropriate to add at this point a few biographical details concerning Kaprekar.

D. R. Kaprekar was born in 1905, in the district of Thana, close to Mumbai. He completed his high school education in 1923, and then joined the Fergusson College (Pune) for his college studies. He became a teacher after graduation, and taught in schools from 1930 till 1962. For more than sixty years he pursued his passion: recreational number theory. He published numerous booklets on his pet interests; many of these were printed at his own expense. He contributed many articles to mathematics journals in the country, and in 1975 his work on the 'Kaprekar constant' became known to readers outside the country through the columns of Martin Gardner in the SCIENTIFIC AMERICAN.

Among the problems that Kaprekar worked on, mention may be made of the Tary-Escot problem, which asks for sets of numbers A and B of equal cardinality, say

$$A = \{a1,a2,\ldots,an\},B = \{b1,b2,\ldots,bn\},$$

such that we have the following relations:

$$a1 + a2 + \cdots + an = b1 + b2 + \cdots + bn,$$

$$a12 + a\,22 + \cdots + a\,n2 = b\,12 + b\,22 + \cdots + b\,n2,$$

$$a13 + a\,23 + \cdots + a\,n3 = b\,13 + b\,23 + \cdots + b\,n3,$$

$$\cdots\cdots = \cdots\cdots$$

$$a1k + a\,2k + \cdots + a\,nk = b\,1k + b\,2k + \cdots + b\,nk,$$

for some given positive integer k. In more compact notation, we must have

$$\sum i=1na\ it = \sum i=1nb\ it$$

for t = 1,2,…,k. The following notation is frequently used by researchers working on this problem: the above relation is written as

$$\{a1,a2,\ldots,aI\} = k\{b1,b2,\ldots,bn\}.$$

Ideally, the sets A and B should be as small as possible. Here are two very interesting Tary-Escot identities found by Kaprekar:

- $\{4, 20, 42, 3, 41, 25\} = 3\{5,21,43,2,40,24\};$.
- $\{6, 32, 45, 8, 45, 39\} = 3\{9,35,48,5,42,36\}$.

In both these examples, remarkably, the digits of the numbers on the right are the same as those of the numbers on the left—but in reverse order!

It is perhaps very appropriate to quote these examples here, as they illustrate the joyful exuberance with which Kaprekar conducted his many number-theoretic researches.

Chapter 8

Self-Born Numbers

8.1 The iteration

In an earlier chapter, we studied the function SSQ(*x*) whose output for an integer input x is the sum of the squares of the digits of x. We now study a similar function which once again is associated with the name of Kaprekar. (For those familiar with Kaprekar's work, this will be seen to be a "typical"Kaprekar iteration.)

For any positive integer x, given in base-10 notation, let SUM (x) denote the sum of the digits of x. For example we have SUM (253) = 2 + 5 + 3 = 10. We study the function g which acts as follows:

$$g(x) = x + SUM(x).$$

In other words, for an input of x, the output is g(x) = x + SUM(x).

Example 1.

- g(25) = 25 + 2 + 5 = 32;
- g(32) = 32 + 8 + 2 = 37;
- g(453) = 453 + 4 + 5 + 3 = 465; and so on.

We now compute, for various *a*'s, the sequences Ka defined as follows:

$$Ka = \langle a;g \rangle.$$

With a = 1, the computations are as set out below:

1,1+1 = 2,2+2 = 4,4+4 = 8,8+8 = 16,16+1+6 = 23,23+2+3 = 28,28+2+8

= 38,38+3+8 = 49,49+4+9 = 62,62+6+2 = 70,70+7+0 = 77,77+7+7 = 91,….

We obtain, therefore:

$$K1 = \langle 1,2,4,8,16,23,28,38,49,62,70,77,91,101,103,107,\ldots\rangle.$$

Obviously 2 generates the same orbit, except for the initial ' 1'. In like manner we obtain:

$$K3 = \langle 3,6,12,15,21,24,30,33,39,51,57,69,84,96,111,\ldots\rangle,$$

$$K5 = \{5,10,11,13,17,25,32,37,47,58,71,79,85,98,115,\ldots\rangle,$$

and so on. It is clear that we can generate infinitely many such sequences, using different seeds.

If we form the union of all such sequences, discarding only their seeds, we obtain a set, say *S*,which consists of all positive integers that can be obtained as output from *g* for *some* choice of input.This is the **RANGE** of g:

$$S = \{g(x) : x \in N\}.$$

What kind of set is S? Does it contain all of **N**?

It certainly contains a good proportion of it. For example, $50 \in S$, because $g(43) = 43 + 4 + 3 = 50$, and $100 \in S$, because $g(86) = 86 + 8 + 6 = 100$. On the other hand, there are many numbers that are not in S.

We claim that 53 is one such number; that is, there is no positive integer x for which x + SUM (x) = 53.

Exercise

8.1.1 Prove this claim.

Proof. We give the proof here, but perhaps you should try the problem on your own before reading on.

Since $SUM(x) \geq 0$ for all positive integers x, the equation x+ SUM (x) = 53 implies that

$$x \leq 53.$$

On the other hand, for positive integers less than or equal to 53, the largest

possible sum-of-digits that we can get is for 49, whose SUM value is 13.Thus:

$$\text{if } x \leq 53, \text{ then SUM } (x) \leq 13.$$

Since x+ SUM (x) = 53, we conclude that

$$x \geq 40.$$

Combining the two inequalities, we conclude that:

$$\text{if } x + \text{SUM } (x) = 53, \text{ then } 40 \leq x \leq 53.$$

We now proceed to construct a table of values of g(x) for $40 \leq x \leq 53$:

x	40	41	42	43	44	45	46	47	48	49	50	51	52	53
$g(x)$	44	46	48	50	52	54	56	58	60	62	55	57	59	61

Note that 53 does not appear in the second row. It follows that $53 \notin S$.

8.2 Exercises

8.2.1 Solve each of the following equations for x:

(i) g(x) = 60;

(ii) g(x) = 148;

(iii) g(x) = 162;

(iv) g(x) = 281;

(v) g(x) = 948.

8.2.2 Using a computer, or otherwise, find all the positive integers below 100 that are not in S.

8.3 Self-born numbers

Kaprekar coined a name for the integers not in S. He called them *self-born*—a beautifully poetic name! The law of information of the sequence of these numbers is unclear; perhaps you could do some research on the problem! Two more examples of such numbers are given below.

- The number 1000000 or 10^6 (i.e., one million) is self-born. (This,

according to Kaprekar, is why millionaires are given so much importance!)

To show why 10^6 is self-born, we argue as we did for the number 53. The steps are, in brief:

* If x+ SUM (x) = 106 then x < 106, ∴ SUM (x) ≤ 54, ∴ x ≥ 106 - 54, i.e., x ≥ 999946. Also, clearly, x ≤ 999999.
* If 999946 ≤ x ≤ 999999 then SUM (x) ≥ 41 (for the lowest SUM for numbers between 999946 and 999999 is 41), ∴x ≤ 999959. Since the largest SUM for numbers that lie between 999946 and 999959 is 50, we see that x ≥ 999950.
* So x lies between 999950 and 999959. One finds quickly that g(x) is odd for all x in this interval. Therefore, none of these *x*-values yield g(x) = 106 (which is even). We deduce that 106 is self-born.

- The well known "cyclic" number 142857 is self-born. (The reader is invited to prove this claim.)

8.4 Junction numbers

Suppose that g(x) = g(y) = N; then under action by g, both x and y yield N. That is, N has two "generators". Numbers with more than one generator are called *junction numbers* by Kaprekar. According to him, the smallest junction number is 101 (with two generators, 91 and 100); the smallest junction number with three generators is

10000000000001,

with 12 zeroes in the "middle" of the number; and the smallest junction number with four generators is

100000000000000000000102,

with 21 zeroes in a block inside the number. (Whew !)

Chapter 9

Self Referential Numbers

9.1 The iteration

The iteration we study now deals with self referential numbers, a curious family of numbers that seem to describe themselves! The best way to describe the iteration is by means of an example.

Example 1. Take any positive integer, e.g., 32. The digits of this number are 2 and 3; there is a single 2 and a single 3. We read out the digits aloud, in increasing order (in this case, 2 first and 3 next) like this: "one two and one three". Next we convert the words of this sentence into digits, i.e., we write ' 1213' (the 'and' is ignored!) and treat this string of digits itself as a number. This number is taken to be the output obtained from the input 32:

$$32 \mapsto 1213.$$

We refer to the function that produces this output as f; thus, f(32) = 1213. Here are a few further examples that show how f acts:

f(12) = 1112 (one 1, one 2"),

f(64) = 1416 (one 4, one 6"),

f(136) = 111316 (one 1, one 3, one 6"),

f(1416) = 211416 (two l's, one 4, one 6"),

f(1117) = 3117 ("three 1's, one 7").

Note that the digits are read off in increasing order, and we make no reference to digits which do not occur in the number.

Does the iteration give rise to any features of interest, e.g., cycles and fixed points? Before going into this question, we first define the term "self referential"

Self Referential Number A fixed point of the function described above is called a self referential number; it satisfies the equation f(x) = x.

Another label we could use for such a number is *self descriptive*, because the number seems to be "describing" itself. However, we shall stick to the term **SELF REFERENTIAL**, because the number seems to 'refer" to itself.

It may seem at first quite hard to find such numbers—indeed, it is not clear that they exist at all; but read on! The examples discussed below should clarify matters.

Example 2. Let x = 22; then f(22) = 22('two 2's'), so 22 is self referential.

Note that 1 is *not* self referential, because f(1) = 11≠1; nor is 0, because f(0) = 10≠0. You are invited to find a few self referential numbers by yourself before proceeding further.

Using the function f to find self referential numbers

We now use the function f as an iterative function—we select some seed number x and then study its orbit ⟨x;f⟩. A curious discovery awaits us: if we try out different seeds, in a majority of the cases the sequence quickly converges to a fixed point of f, that is, to a self referential number.

Example 3. We examine the sequence ⟨1;f⟩.

$$f(1) = 11,$$
$$f(11) = 21,$$
$$f(21) = 1112,$$
$$f(1112) = 3112,$$
$$f(3112) = 211312,$$
$$f(211312) = 312213,$$
$$f(312213) = 212223,$$

f(212223) = 114213,

f(114213) = 31121314,

f(31121314) = 41122314,

f(41122314) = 31221324,

f(31221324) = 21322314,

f(21322314) = 21322314,

and that's it! We have:

$$f(21322314) = 21322314,$$

so 21322314 is self referential. What a surprise!

9.2 Exercises

9.2.1 Using the seeds x = 3,x = 4 and x = 9, see if the sequences ⟨x;f⟩ give rise to self referential numbers.

9.2.2 Repeat the exercise with the seeds x = 20,x = 25,x = 95. Tabulate your results.

9.2.3 Repeat the experiment with the seeds x = 40,x = 50. (You may find the results rather surprising.)

9.2.4 Find other seeds that behave like 40 and 95.
A lot of experimentation will be needed to answer this question! See if you can get a computer to do the work for you. This itself is no simple feat—it requires the writing of a good computer program.

9.2.5 [*] Challenge ! Explain why the phenomenon occurs at all, i.e., why for any seed x, the sequence ⟨x;f⟩ converges to a self referential number or a self referential cycle. (See Chapter 14 for some ideas on how this might be done.)

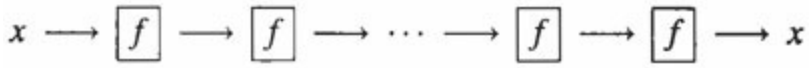
Chapter 10

Cyclical Functions

10.1 Iterations on functions

We now look for repeating cycles of a rather different variety, exhibited not by some specific numbers but *by the functions themselves*. As stated earlier, a function is essentially a machine that converts an input into an output. What would 'cyclic behaviour' mean in this context? Simply this: *the function must be such that the action of continually feeding its output back as input into itself must produce, after some steps, the same number with which we began*. Moreover, this must happen regardless of the choice of initial seed. We shall refer to such a function as a *cyclical function*.

Visually, one might represent this as follows:



Here the "…" means a certain number of f′s. The point is that when the input, x, is passed through all these machines, after some steps we obtain the original value x once again. *This must hold no matter which input we start with*.

Do there exist cyclical functions at all?

The surprising answer is that there are very many such functions; indeed, infinitely many!

Example 1. Consider the function *R* (*'R'* for 'reciprocal') defined thus for $x \neq 0$:

$$R(x) = 1\ x.$$

Thus R yields 2 from 1⁄2; and now if 1/2 is fed back into R, we get 2, because

$$R\ 1\ 2 = 1\ 1⁄2 = 2.$$

Visually, we have:

$$2\mapsto 1\ 2\mapsto 2.$$

This will of course happen for any non-zero input *x*:

$$R(x)=1x, \therefore R1x=11/x=x.$$

We write this more compactly as follows:

$$R(R(x)) = x \quad \text{for all } x\neq 0.$$

This is read aloud as "*R* of R of x equals x for all non-zero x." (Note that the phrase 'for non-zero x' must be included: the function would not 'know' what to do with an input of 0, because 1/0 is an undefined quantity.)

Still more compact is the following symbolism:

$$R \circ R = I.$$

By '$R \circ R$' (read this as '*R* of *R*′) we mean that *R* is applied twice over, and by '*I*' we mean the *identity function*—the function for which the output always equals the input ($I(x) = x$ for all values of x).

What are the cycles of *R*? It is easy to see that the only fixed points are 1 and - 1, because the equation $1⁄x = x$ leads to $x^2 = 1$, whose solution set is {1, -1}. We conclude that for the function $R(x) = 1⁄x$,

- each non-zero number other than 1 or-l belongs to a cycle of length *2*; and.
- there are precisely two l-cycles, namely ⟨1⟩ and ⟨-1⟩.

Example 2. Consider the function g given by

$$g(x) = 1 - x.$$

Observe that g takes 2 to - 1 (because 1 - 2 = -1), and –1 back to 2 (because 1 - (-1) = 2):

$$2\mapsto 1 - 2 = -1\mapsto 1 - (-1) = 2.$$

The same thing happens for any input x, because

$$x \mapsto 1 - x \mapsto 1 - (1 - x) = x,$$

or, written differently,

$g(g(x)) = 1 - g(x) = 1 - (1 - x) = x.$

In terms of the new symbolism, $g \circ g = I$.

Note that g has just one 1-cycle; for if $g(x) = x$ then $1 - x = x$, $\therefore x = 1/2$. Thus, apart from a solitary 1-cycle, namely $\langle 1/2 \rangle$, every other number belongs to a 2-cycle.

For a still simpler example, consider the function h given by $h(x) = -x$. Here $\langle 0 \rangle$ is a l-cycle, because $- 0 = 0$, and it is the only 1-cycle (please verify this on your own). As earlier, the relation $h(h(x)) = x$ is true for every value of x.

Exercises

10.1.1 Verify that $\langle 0 \rangle$ is the only l-cycle. of the function h given by $h(x) = -x$.

10.1.2 Produce two more 2-cycle functions.

10.1.3 Produce a 2-cycle function k with no fixed points. Here we must have

$$k(x) \neq x \quad \text{for every x,}$$

but on the other hand,

$$k(k(x)) = x \quad \text{for every x.}$$

(Warning: This is a tricky problem !)

10.1.4 Consider a function h defined as follows:

$$h(x) = 1 - 1 x \quad \text{for } x \neq 0, x \neq 1.$$

i. Why is it necessary to impose the two restrictions, $x \neq 0, x \neq 1$?

ii. What cycles are produced by this function? Experiment with a few numbers and make a guess based on your results. Verify your claim using simple algebra.

10.1.5 Find the cycle behaviour of the function m defined thus:

$$m(x) = x\,x - 1 \quad \text{for } x \neq 0, x \neq -1.$$

10.1.6 Repeat the analysis for the function n, where

$$n(x) = 1\,1 - x \quad \text{for } x \neq 0, x \neq -1.$$

The results of these exercises are indeed very curious; we see examples of cyclical functions with cycle lengths of 2, 3 and 6. How might we go about finding more such functions?

10.2 Constructing cyclical functions

It can be quite tricky to construct a cyclical function with a specified cycle length! For instance, how would you construct a function with a cycle length of 4 or 5? The latter is quite challenging to find if one considers only "nice" functions, that is, functions defined by simple algebraic expressions.

Here is another example of a 2-cycle function p:

$$p(x) = x - 1 \text{if} [x] \text{is odd } x + 1 \text{ i f } [x] \text{i s e v e n .}$$

Here [x] (denoted by int(x) ’ in *BASIC* and by. floor (x) in other languages) is the largest integer less than or equal to x. It is referred to by computer science students as the **FLOOR FUNCTION**. (Would you consider this a "nice" function?.)

Example 3. $[5.9] = 5, [8.71] = 8, [-8.71] = -9, [\pi] = 3.$

Remark. Note that we have defined p by means of a rule rather than through a formula. Actually, rules and formulas are not very different from one another! Both in essence tell you what to do with a given input. Indeed, in mathematics, the words "formula" , "rule" and "algorithm" all convey much the same meaning.

Here is an example to show how p acts. We.take an input value, say 7.3. Since [7.3] = 7, which is odd, we have:

$$p(7.3) = 7.3 - 1 = 6.3,$$

and since [6.3] = 6, which is even, we have:

$$p(6.3) = 6.3 + 1 = 7.3.$$

Thus we have

$$7.3 \mapsto 6.3 \mapsto 7.3,$$

or:

$$p \circ p(7.3) = 7.3,$$

and this fits in with how we wanted p to 'behave'.

Exercises

10.2.1 Compute the values of p(3.1),p(-7.3),p(0).

10.2.2 Show that p has no fixed point (i.e., $p(x) \neq x$ for all x), but $p \circ p = I$.

10.2.3 Find all the values of x for which p(x) = 3.3.

10.2.4 Use the ideas presented above to construct a 5-cycle function f(that is, we have $f \circ f \circ f \circ f \circ f = I$, which may also be written as f(5) = I, but $f \neq I$).

10.2.5 Show how one can construct an n-cycle function for any given positive integer n.

10.3 Fractional iterates

An interesting concept that now arises is that of "iterated square root of a function". We illustrate the concept with an example.

If f(x) = x - 1, then f(f(x)) = (x - 1) - 1 = x - 2. Writing h(x) = x - 2, we have f(f(x)) = h(x) for all x, that is:

$$f \circ f = h.$$

Writing $f \circ f$ as f(2) for short, we have:

$$f(2) = h.$$

Thus f can be called an "iterated square root" of h, and we can write $f = h(1/2)$.

Remark. It is important not to confuse this notation with the more common one of squaring a number or expression; that is, with expressions such as $5^2 = 25$. The notation here refers to an *operation* being applied twice

over. That is why we have used the notation f(2) and not f2, which could suggest the squaring operation. However, owing to the possibility of confusion, we shall not employ this symbolism very often.

Iterated Square Root of a Function Given a function a that maps **R** into **R** if a function b (also defined on **R**) exists, such that b ◦ b = a (i.e., b(b(x)) = a(x) for all values of x), then b is called an iterated square root of a.

Note that we have not said '*the* iterated square root'; it is possible for a function to have more than one iterated square root, or to have no iterated square root at all.

Example 4. Let a(x) = 1 1 - x and b(x) = 1 -1 x; then b ◦ b = a.

To see why, we apply the function b twice over:

$$x \mapsto 1\ -1\ x = x - 1\ x \mapsto 1 - x\ x - 1 = -\ 1\ x - 1 = 1\ 1 - x = a(x),$$

so b is indeed an iterated square root of a.

Example 5. The identity function I(x) = x has as iterated square roots each of the following functions:

$$b1(x) = 1\ x, b2(x) = 1 - x, b3(x) = -x.$$

Are there any other iterated square roots of I(x)?

Exercises

10.3.1 Verify the statement made in Example 5, above.

10.3.2 Determine possible iterated square roots for each of the following functions. Assume that the domain in each case is the set of real numbers.

i. a(x) = x + 1

ii. a(x) = x + 2

iii. a(x) = 4x

iv. a(x) = 2x

10.3.3 What might be a good definition for "iterated cube root" of a function?

Does the function f, where f(x) = -x, have an iterated cube root?

10.4 Iterated square root of the negative identity function

We now describe a rather tricky procedure for constructing an iterated square root of the "negative identity" function J(x) given by:

$$J(x) = -x.$$

Let the iterated square root be called b. We show below how to compute b(x) for any given x; this will serve to define b.

1. If x = 0, we assign to b the value 0, that is, b(0) = 0.
2. If x > 0, we find the unique integer n for which

   $$10n \leq x < 10n+1,$$

   that is, we find the two successive powers of 10 between which x lies. The value of b(x) is now calculated as shown below:

   $$bx=10xifnis\ even-x/10ifnis\ odd.$$

3. If x < 0, we go through the procedure described above, but with - x in place of x. That is, we find the unique integer n for which

   $$10\text{-}n \leq -x < 10\text{-}n+1,$$

   and the value of b(x) is calculated thus:

   $$b(x) = 10x\ ifniseven\ -\ x/10\ if\ nis\ odd\ .$$

Then b satisfies the relation b(b (x)) = -x for all x, i.e., b(2) = -I. (Note that this implies the relation b(4) = I.) To see why, we experiment with a few numbers.

Example 6. Let x = 2. Then x > 0; since 100 < 2 < 101 we get n = 0, an even number; so b(2) = 10 × 2 = 20.

Next, 20 > 0; since 101 < 20< 102 we get n = 1, an odd number; so b(20) = -20/10= -2. Therefore b(b(2)) = -2, in agreement with the claim made. Note the route taken by the initial input of 2:

$$2\mapsto 20\mapsto -2.$$

10.5 Exercises

10.5.1 Show that the function b described abovee satisfies the relation b(b(x)) = -x for all real values of x.

10.5.2 Find iterated square roots for each of the following functions:

i. a(x) = x2; domain: all real numbers.

ii. a(x) = x 2x + 1; domain: all real numbers x≠ -1 2.

iii. a(x) = 1 x; domain: all non-zero real numbers.

10.5.3 Let Z denote the set of integers. Construct a function f from Z into Z such that f(4)(x) = -x for all x in Z; that is,

$$f \circ f \circ f \circ f(x) = -x \text{forallx} \in Z.$$

Comment: The corresponding problem for R rather than Z is much more difficult.

10.5.4 [*] In this problem R refers to the set of real numbers.

Consider the function g with domain R, given by

$$g(x) = x2 - 2.$$

Prove that there does not exist any function f from R into R such that

$$f \circ f = g,$$

that is, such that for all real numbers x, we have f(f(x)) = x2 - 2.

This means that g has no iterated square root function if we restrict ourselves to functions from R into R.

10.5.5 [*] In this problem N refers to the set of positive integers.

Consider the function g with domain N, given by

$$g(n) = n + 1987.$$

Prove that there does not exist any function f from N into N such that

$$f \circ f = g,$$

i.e., such that for all positive integers n we have f(f(n)) = n + 1987.

This means that g has no iterated square root function if we restrict ourselves to functions from N into N.

(This problem is from the International Mathematical Olympiad held in Cuba in 1987.)

Chapter 11

A Miscellany–I

This chapter and the following one present several miscellaneous iterations and study the rich mathematical patterns which arise from them.

11.1 Appetizers

11.1.1 For each function f listed below, compute the first ten terms of the sequences $\langle x;f \rangle$ produced by the seeds x = 1,x = 2,x = 10. Suggest what the limiting behaviour of each sequence might be.

i. f(x) = 1 + x 2.

ii. f(x) = 1 + x.

iii. f(x) = 1 + 2x.

iv. f(x) = 1 + 1 x2.

v. f(x) = 1000 + x3.

11.1.2 [*] Try proving or disproving the conjectures made above.

11.1.3 The function f that we study in this problem acts on number pairs rather than single numbers, and its action is as follows:

$$f(x,y) = y, y + 1\ x\ .$$

For instance:

$$f(1,2) = (2,3),f(10,20) = 20,21\ 20.$$

To make sure that the arithmetic is "proper", we impose the following

restrictions on x,y:

$$x \neq 0, y \neq -1.$$

(To see why the restrictions are needed, compute the orbits of (1,0) and (1, - 1); what do you find?)

Compute the orbits of the seeds (1, 2), (1, 3), (1, 4) and (1, 5). What do you find? Based on your observations, formulate a hypothesis, then check the hypothesis using simple algebra.

11.1.4 Should any additional restrictions be placed on the seed (x,y) to ensure that no "improper arithmetic" occurs during the iteration? (One possibility is the condition $x,y > 0$. This may be too strong, however.) See if you can find all the necessary restrictions.

11.2 The Collatz iteration

Here is a particularly famous iteration whose peculiarity is that it is easy to define but extremely difficult to analyse. Its behaviour is so complex that its cycle structure has yet to be unravelled! The iteration uses the "Collatz" function C (so-named after Lothar Collatz, the mathematician who first studied it) which acts on positive integers as follows:

$$C(x) = x/2 \text{ if } x \text{ is even } 3x + 1 \text{ if } x \text{ is odd .}$$

Example 1 .

- $C(10) = 5$, since 10 is even, and $C(3) = 10$, since 3 is odd.
- The orbit $\langle 10;C \rangle$ is found to be

  $$\langle 10,5,16,8,4,2,1,4,2,1,4,2,1,\ldots \rangle.$$

  Observe that we have reached the 3-cycle $\langle 4,2,1 \rangle$.
- The orbit $\langle 23;C \rangle$ is found to be

  $$\langle 23,70,35,106,53,160,80,40,20,10,5,16,8,4,2,1,4,2,1\ldots. \rangle.$$

  We have reached the very same 3-cycle, $\langle 4,2,1 \rangle$.
- If we start with a seed of 47, a surprise is in store for us. The 3-cycle $\langle 4,2,1 \rangle$ is reached yet again, but after a very long time indeed:

⟨ 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364,182, 91, 274, 137, 412, 206, 103, 310,155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1790, 890, 445, 1336, 668, 334,167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077 ,9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, … ⟩ (whew!).

How might we go about listing the cycles of the Collatz function? Before plunging into the investigation, read the following rather surprising news:

The set of cycles of C *is not known at the present time*!

There is strong numerical evidence, however, that ⟨4,2,1⟩ is the *only* cycle, and moreover that every orbit eventually reaches this cycle. Another way of expressing this is the following: for any positive integer n, there exists a value of k > 0 such that

$$C(k)(n) = 1,$$

where, of course, C(k) denotes the *k*-fold iterate of C. For some seeds the orbit proceeds for millions of terms before reaching the cycle—the situation can be much 'worse' than for 47. As of now, however, no one has found a seed for which the cycle ⟨4, 2, 1 ⟩ is not reached, nor has anyone proved that this will always happen.

Exercise

11.2.1 Find the seed number in the set {1, 2, *3*, …, 1000} that takes the longest to reach the cycle ⟨4, *2*, 1 ⟩. (Use a computer!)

Brief history of the Collatz problem

The Collatz problem, also known as the "3x + 1" problem, and by other names too, has acquired great notoriety in the community of mathematicians. It seems to have first surfaced in the years immediately preceding World War II, but it was in the 1950s that it grew in fame—possibly because it was

mentioned during a public lecture by the well known geometer H. S. M. Coxeter; interestingly, he offered a prize of $50 for its first valid proof, and a prize of $100 for the first counterexample! Needless to say, the prizes have not been claimed by anyone till now. The great mathematician Paul Erdős is reported to have made the following comment on the problem: "*Mathematics is not yet ripe enough for such problems.*"

Collatz's cousins

The 3x + 1 problem has a few 'cousins'—problems which bear a close resemblance to the Collatz problem but, strangely, are very different with regard to their intrinsic difficulty. We describe here one such 'cousin'. Let f be the function defined on N thus: for x ∈N,

$$f(x) = x/2 \text{ if } x \equiv 0 \pmod 4,\ 3x + 1 \text{ if } x \equiv 1 \pmod 4,\ 2x - 1 \text{ if } x \equiv 2 \pmod 4,\ (x + 1)/4 \text{ if } x \equiv 3 \pmod 4.$$

The definition certainly looks rather more complicated than the original one! Here are two typical orbits produced⁻ when we iterate f on **N**.

$$\langle 11; f \rangle = \langle 11,3,1,4,2,3,1,4,2,3,1,\ldots \rangle,$$

$$\langle 5; f \rangle = \langle 5,16,8,4,2,3,1,4,2,3,1,\ldots \rangle.$$

Observe that in both cases we reach the 4-cycle ⟨4, 2, 3, 1⟩. Further experimentation suggests that this is the only f-cycle; and, as with the Collatz iteration, every orbit seems to reach this cycle. This is indeed so, but it is relatively easy to prove. Here is how we may proceed.

We first check what happens when we iterate f on seeds of the forms 4k,4k + 1,4k + 2,4k + 3. We argue as follows.

- If x is even, then x is of one of the forms 4k,4k + 2.
- If x = 4k, then f(x) = 2k = x/2.
- If x = 4k + 2, then f(x) = 8k + 3, which is of the form 3 (mod4), therefore

$$f(2)(x) = 8k + 3 + 1\ 4 = 2k + 1; \text{ i.e., } f(2)(x) = x\ 2.$$

- So if x is an even number, then either f(1)(x) = x/2 or f(2)(x)=x2. That is, in either one step or in two steps the orbit of xre aches x/2.

- If x is odd, then x is of one of the forms 4k + 1,4k + 3.
- If x = 4k + 1, then f(x) = 12k + 4, which is of the form 0(mod4), implying that

  $$f(2)(x) = f(12k + 4) = 6k + 2,$$

  and since 6k + 2 is an even number we conclude that either f(3)(x) = 3k + 1, or f(4)(x) = 3k + 1. (The former happens if 6k + 2 is 0(mod4), the latter if 6k + 2 ≡ 2(mod4).) So if x = 4k + 1 then in either three steps or in four steps the orbit of x reaches (3x + 1)⁄4.
- Finally, if x = 4k + 3 then f(x) = k + 1 = (x + 1)⁄4.

It follows from this that the cycles of f may be matched in a one-to-one manner with the cycles of the following function g which also acts on N: for x ∈N,

g(x) = x⁄2 ifx ≡ 0(mod2), (3x + 1)⁄4ifx ≡ 1(mod4), (x - 1)⁄4 ifx ≡ 2(mod4),

Note the subtle point: f *is not the same as* g, *but* lts *cycles correspond exactly to those of* g; *its long-term behaviour is similar to that of* g.

Now we examine the definition of g. It is clear that if x is even or if x ≡ 3(mod4) then f(x) < x (the inequality is strict); and if x ≡ 2(mod4), then since

$$3x + 1\ 4 \leq x, \text{ with equality only when } x = 1,$$

we have g(x) ≤ x. We deduce from this that 1 is the sole fixed point of f, and that g(x) < x for all x > 1. The immediate deduction from this is that for any seed x ∈N, the sequence

$$x,g(x),g(g(x)),g(g(g(x))),\ldots$$

strictly decreases till it reaches the fixed point 1, after which it stays constant. This means that the *only* cycle of g is the l-cycle ⟨1⟩. This in tum implies that the only cycle of the original function f is {1,4, 2, *3*}. And since every orbit of g reaches 1, every orbit of f must reach the cycle ⟨ 1, 4, 2, 3⟩. So our conjecture about the cycles of f is resolved. Indeed, it yielded rather easily to analysis. But this is not the case with the Collatz iteration.

References

Here are a few references to the Collatz problem.

1. Coxeter, H S M. *Cyclic Sequences and Frieze Patterns* (The Fourth Felix Behrend Memorial Lecture), 1970.
2. Gardner, Martin. *Mathematical Games*, Scientific American, 226 (June 1972) 114–118.
3. Lagarias, J. *The* 3x + 1 *Problem and its Generalizations*, American Mathematical Monthly, Volume 92, Number 1 (January 1985), 3-22.
4. Trigg, C L, Dodge, C W and Meyers, L F. *Comments on Problem 133*, Eureka (now Crux Mathematicorum) Volume *2*, Number 7 (August-September 1976), 144-150.

11.3 A two-digit iteration

In this section, we restrict our attention to the set

$$S = \{0,1,2,3,\ldots,98,99\}.$$

The function f that we study has S as its domain and is defined thus: given an integer n in S, we write n as 10a + b, where a,b are the digits of n (note that a = 0 for a l-digit number), and then define f(n) to be $|a2 - b2|$, where || refers to absolute value.

Example 2. f(31) = $|32 - 12| = 8$;f(63) = $|62 - 32| = 27$;f(7) = $|02 - 72| = 49$; etc.

Note that f(n) $\in$ S for all n $\in$ S (for f(n) < 99 for all n $\in$ S; indeed, the largest value taken by f on S is clearly f(90) = 81), so it makes sense to iterate f on the set S:. We now proceed to determine the cycles of f, using an idea which we have not discussed till now.

Computing the cycles of f

We start by computing the range of f, written f(S) for short, by letting f act on the entire set S and listing the numbers thus produced. The computation can be done using the following computer program (in BASIC).

```
REM A Two Digit Iteration
```

```
FOR a = 0 to 9
FOR b = 0 to 9
    X = 10*a + b : y = ABS (a∧2 - b∧2)
    PRINT x, y
NEXT b
NEXT a
```

The set of y's is the range we seek. After removing duplicated elements and sorting the remaining numbers (i.e., arranging them in ascending order), we obtain the set displayed below.

Range (f) = {0,1,3,4,5,7,8,9,11,12,13,15,16,17,20,21,24,25,27,28,32,33, 35,36,39,40,45,48,49,55,56,60,63,64,65,72,77,80,81}.

This set is f(S); we call it S1 for short. Observe that S1 has fewer elements than S. (Check: |S| = 100, whereas |S1| = 39.)

We now let f act on S1, list the numbers produced, and once again go through the exercise of removing duplicated elements and sorting the leftover numbers. The result is the set S2 = f(S1) displayed below.

S2 = {0,1,3,4,5,8,9,11,12,16,20,21,24,25,27,35,36,45,48,49,60,63,64,65,72,8

Observe that S2 has fewer elements than S1. (Check: |S2| = 26, whereas |S1| = 39.)

We continue in this manner and compute the sets

$$S3 = f(S2), S4 = f(S3), S5 = f(S4), \ldots.$$

Initially we observe a steady contraction in size: |S3| = 20,|S4| = 15,|S5| = 12 and |S6| = 10:

S3 = {0,1,3,4,9,11,12,16,20,21,25,27,35,36,45,48,63,64,65,81},

S4 = {0,1,3,4,9,11,16,20,21,27,35,45,48,63,81},

S5 = {0,1,3,4,9,16,27,35,45,48,63,81},

S6 = {0,1,9,16,27,35,45,48,63,81},

S7 = {0,1,9,16,27,35,45,48,63,81}.

But now observe that S7 is identical to S6 in all its elements! This being so, we do not need to proceed any further; no further 'contraction' can possibly take place. Writing T = S7, we see. that f(T) = T. So T is a "fixed set".

We now claim that *the cycles of* f *are all to be found within the elements of* T. Therefore to list the cycles of f we only need to list the f-values of all the elements in T:

n	0	1	9	16	27	35	45	48	63	81
$f(n)$	0	1	81	35	45	16	9	48	27	63

Now, by hooking up appropriate pairs of numbers, the cycles of f. are immediately revealed:

⟨0⟩;⟨1⟩;⟨9,81,63,27,45⟩;⟨16,35⟩;⟨48⟩.

So fhas three l-cycles, a 2-cycle and, surprisingly, a 5-cycle.

Exercises

11.3.1 Why does this technique (illustrated above) "work"? Why does it yield all the cycles of f?

11.3.2 Try the technique on the function g acting on the set S = {1,2,3, …,98,99} (the set of positive integers less than 100), and defined t1hus: if n = 10a + b, where a,b are the digits of n, then

$$g(n) = |a2 - b2| + |a - b|.$$

Does it yield all the cycles as promised? (It does !—there are two l-cycles and one 9-cycle.)

11.4 Amicable numbers

In this section, we dwell briefly on a concept first studied by the Greeks, that of *perfectness* of a number. We srm by defining a function D that acts on the

positive integers N thus: D(1) = 1, and if n is greater than 1, then D(n) = the sum of the proper divisors of n. (We include 1 as a proper divisor; of course, n itself is not a proper divisor of, itself.) For example,

D(6) = 1 + 2 + 3 = 6,D(10) = 1 + 2 + 5 = 8,D(20) = 1 + 2 + 4 + 5 + 10 = 22.

Let n be called **PERFECT** if D(n) = n. Thus, 6 is perfect, because D(6) = 6. Note that our definition would require 1 too to be called perfect, as D(1) = 1; however, 1 is not ordinarily regarded as perfect, and we follow this convention.

The Greeks were intrigued by the idea of perfectness, and Euclid's great book, THE ELEMENTS, has the following very pretty result:

> *Let* P *be a prime number of the form* 2k - 1, *then the number* 1 2P(P + 1) *is perfect.*

The proof is given in the book, and in an elegant form!

In modem times, primes of the form 2k - 1 have come to be known as Mersenne primes, after the French monk Father Marin Mersenne (1588–1648; he was also a number theorist!). It is easy to give examples of such primes, and of the perfect numbers associated with them:

a. k = 2 yields the Mersenne prime P = 3 and the perfect number 3 × 2 = 6.
b. k = 3 yields the Mersenne prime P = 7 and the perfect number 7 × 4 = 28 (a quick check: 28 has as divisors the numbers 1, 2, 4, 7 and 14; and 1 + 2 + 4 + 7 + 14 = 28).
c. k = 5 yields the Mersenne prime P = 31 and the perfect number *3* 1 × 16 = 496. (Please verify for yourself that 496 is perfect.)

Note that for 2k - 1 to be prime, k itself must be prime (but, unfortunately, this condition does not suffice).

From perfectness to amicability; ibn Qorrah's rule

If we iterate the sum-of-proper-divisors function, D, on the natural numbers N, then the l-cycles correspond to the perfect numbers. Are there larger cycles? The Greeks had an amusing term for pairs of numbers that form a 2-

cycle: they called them an AMICABLE PAIR, because the sum of the proper divisors of each number is equal to the other number. Pythagoras went so far as to declare that friends ought to relate to one another in the same manner that amicable numbers do! The example quoted by the Greeks w *a*s the pair {220, 284}:

$$D(220) = 1 + 2 + 4 + 5 + 10 + 11 + 22 + 44 + 55 + 110 = 284,$$

$$D(284) = 1 + 2 + 4 + 71 + 142 = 220.$$

This was probably the only example the Greeks had of amicable numbers, and it was left to future generations of mathematicians to list more such pairs. The great mathematician Leonhard Euler (1707-1783; the name is pronounced as "Oiler") listed no less than sixty pairs! The Arabian mathematician ibn Qorrah gave the following elegant rule to determine further pairs of amicable numbers:

Let $n \in N$ *be such that the three numbers*

$$a = 3 \cdot 2n - 1, b = 2a + 1, c = ab + a + b$$

are all prime; *let* $M = 2n+1$ *ab*, $N = 2n+1c$. *Then* {M,N} *is an amicable pair*.

Example 3. With $n = 1$, we obtain $a = 5, b = 11, c = 71$, and these all happen to be prime. These give $M = 4 \times 55 = 220, N = 4 \times 71 = 284$, and we obtain the amicable pair {*220*, 284} listed above.

Example 4. With $n = 2$, we obtain $a = 11, b = 23, c = 287$, but 287 is not prime ($287 = 7 \times 41$), so we get nothing here.

Example 5. With $n = 3$, we obtain $a = 23, b = 47, c = 1151$, and these are once again all prime. These give $M = 16 \times 1081 = 17296, N = 16 \times 1151 = 18416$, and we obtain the amicable pair {17296, 18416}.

Checking for amicability

To check that ⟨17296, 18416⟩ is a 2-cycle would seem tedious, but the task is simplified greatly by using the following rule. *Let* $n = pa \times qb \times rc \times \cdots$. *where* p,q,r, . . . *are the distinct primes that divide* n, *and* a,b,c *are positive integers. Let*

$$P = 1 + p + p2 + p3 + \cdots + pa,$$

$$Q = 1 + q + q2 + q3 + \cdots + qb,$$

$$R = 1 + r + r2 + r3 + \cdots + rc,$$

and so on. Then $D(n) = (P \times Q \times R \times \cdots) - n$.

Example 6 To compute D(1728), observe that the prime factorization of 1728 is given by $1728 = 123 = 26 \times 33$. We now have:

$$P = 1 + 2 + 22 + \cdots + 26 = 127,$$

$$Q = 1 + 3 + 32 + 33 + 34 = 121,$$

therefore $D(1728) = (127 \times 121) - 1728 = 1\ 3639$.

We are now in a position to verify that ⟨17296, 18416⟩ is a 2-cycle. Since $17296 = 24 \times 23 \times 47$ and $18416 = 24 \times 1151$, we obtain:

$$D(17296) = (31 \times 24 \times 48) - 17296 = 18416,$$

$$D(18416) = (31 \times 1152) - 18416 = 17296,$$

and the verification is complete.

Unfortunately there are not too many values of n for which the numbers a,b,c as given by ibn Qorrah are all prime. A computer search conducted for values of n below 100 reveals that the only successful values *are* n = 1,3 and 6. We have already listed the amicable pairs produced by n = 1 and 3; the one corresponding to n = 6 is {9363584, 9437056}.

Searching for more 2-cycles

Searching for 2-cycles empirically is not easy, but the technique presented in the last section (Two digit ireration") suggests a way out. We define a function a that operates on *sets* and is computed thus. Let X be a set of positive integers. We first compute D(X), the set of D-values of the integers in X. For example,

$$D(\{6,10.20\}) = \{6,8,22\};$$

$$D(\{10,20,30,40\}) = \{8,22,42,50\}.$$

The set a(X) is now computed as follows:

$$a(X) = D(X) \cap X.$$

Example 7. To compute a(T), where

$$T = \{1,2,3,4,5,6,7,8,9,10\},$$

we first compute D(T); we get:

$$D(T) = \{1,3,4,6,7,8\}$$

(the duplicates have been removed). Since D(T) is a subset of T, we get D(T) $\cap$ T = D(T), and so

$$a(T) = \{1,3,4,6,7,8\}.$$

As usual, we define a(2) = a $\circ$ a,a(3) = a $\circ$ a $\circ$ a,… ; that is, a(k) is the kth iterate of a. (a(1) is the same *a*s a.) If we iterate a on T, here is what we find:

$$a(1)(T) = \{1,3,4,6,7,8\},$$

$$a(2)(T) = \{1,3,6,7\},$$

$$a(3)(T) = \{1,6\},$$

$$a(4)(T) = \{1,6\}.$$

Since no change has occurred at the last step, that is, a(3)(T) = a(4)(T), it follows that a({1,6}) = {1,6}. *This means that each element in the set* {1, *6*} *is part of a cycle*. We already know this to be true, since D(1) = 1,D(6) = 6.

So the trick is to iterate a on a very large set, and check whether after a suitable number of iterations we obtain a set which remains unchanged after action by a; in other words, a "D-invariant set" (one which undergoes no change when D acts on it). This is excessively. tedious to do by hand, but easily accomplished by using a computer algebra package such as***MATHEMATICA.*** We start with the set

$$\{ 1, 2, 3 , 4, \ldots, 99999, 100000 \} ,$$

and apply a repeatedly. ***MATHEMATICA*** ultimately yields the following invariant set S:

{1, 6, 28, 220, 284, 496, 1184, 1210, 2620, 2924, 5020, 5564, 6232, 6368, 8128, 10744, 10856, 12285, 12496, 14264, 14288, 14536, 14595, 15472, 17296, 18416, 63020, 66928, 66992, 67095, 69615, 71145, 76084,

79750, 87633, 88730 } .

We may infer now that each element of S is pan of a cycle; and indeed, we do find cycles in abundance! There are five l-cycles, corresponding to 1 and to the perfect numbers 6, 28, 496 and 8128. Next, there are numerous 2-cycles, and these correspond to pairs of amicable numbers:

{220, 284}, {1184, 1210}, {5564, 5020},
{*6232*, 6368}, {10744, 10856}, {12285, 14595},
{17296, 18416}, {63020, 76084}, {66928, 66992},
{67095, 71145}, {69615, 87633}, {79750, 88730}.

There is even a 5-cycle!—

⟨12496,14288,15472,14536,14264⟩.

Note that no 3-cycle or 4-cycle has, been found; perhaps we need to sm with a much larger set (say all the integers from 1 till 106) in order to find such cycles.

Exercises

11.4.1 What are the integers n for which D(n) = 1?

11.4.2 Verify that the rule for computing D(n) yields the correct values for n = 10,20,30,40 and 50.

11.4.3 Why does the rule work?

11.4.4 Define the terms *abundant* and *deficient* thus: if D(n) exceeds n, then n is 'abundant', and if D(n) is less than n, then n is 'deficient'.

Show that if n is abundant, then so is use this to conclude that the number of abundant integers is infinite.

11.4.5 Are there any integers n for which D(n) = 2?

11.4.6 Prove Euclid's assertion: *if* P *is a prime number of the form* 2k - 1, *then* P(P + 1)⁄2 *is perfect*.

11.4.7 Prove the following stronger assertion, due to Euler.

Every even perfect number is of the from 1 2P(P + 1) *where P is a*

mersenne prime, i.e., *a prime number of the from* 2k - 1.

The two results (Euclid's and Euler's) acting in tandem show that there is a one-to-one correspondence between the Mersenne primes and the even perfect numbers.

11.4.8 Verify that {6232, 6368} is an amicable pair.

11.4.9 Verify that {9363584, 9437056} is amicable. You will need the following factorizations:

a. 9363584 = 2 7 × 191 × 383;

b. 9437056 = 27 × 73727.

11.4.10 Prove ibn Qorrah's claim: that the pair {2n+1ab,2n+1c} is amicable if the three numbers a = 3 · 2n - 1,b = 2a + 1 and c = ab + a + b are all primes.

Proof. We give the proof here, but you should try the problem on your own before reading on. We shall establish somewhat more.

Let M = 2kab and N = 2kc, where k is non-negative and a,b,c are odd primes with a≠b. We seek conditions on k,a,b,c that make {M,N} an amicable pair. Using the rule given earlier, we obtain:

$$D(M) = 2k+1 - 1(a + 1)(b + 1) - 2kab,$$

$$D(N) = 2k+1 - 1(c + 1) - 2kC .$$

Therefore the relations D(M) = N,D(N) = M hold if and only if

$$2k+1 - 1(a + 1)(b + 1) = 2k(ab + c),$$

$$2k+1 - 1(c + 1) = 2k(ab + c).$$

From these we deduce that c + 1 = (a + 1)(b + 1), that is, c = ab + a + b. *This condition is forced on us.* Assuming that this holds, we see that {M,N} is amicable if and only if

$$2k+1 - 1(a + 1)(b + 1) = 2k(2ab + a + b).$$

After a bit of manipulation, this simplifies to:

$$ab - 1\ a + b + 2 = 2k - 1.$$

So if we find prime numbers a,b and c = ab + a + b for which this holds, then we obtain the amicable pair {2kab,2kc · }.

Progress, is achieved by imposing an extra condition: b = 2a + 1. We then get a + b + 2 = 3(a + 1) and ab - 1 = a(2a + 1) - 1 = 2a2 + a - 1, which factorizes:

$$2a2 + a - 1 = (2a - 1)(a + 1).$$

The factor (a + 1) cancels out, and we obtain (2a - 1)⁄3 = 2k - 1, which further simplifies to

$$a = 3 \cdot 2k\text{-}1 - 1.$$

Thus we seek positive integers k such that a = 3 · 2k-1 - 1,b = 2a + 1 and c = ab + a + b are all prime. *This is just* ibn *Qorrah's rule.* ❑

11.5 The Fibonacci sequence

Let a function f be defined thus for non-zero real numbers:

$$f(x) = 1 + 1\ x,$$

and consider the orbit ⟨1;f⟩. The computations are done as shown below:

$$f(1) = 1 + 1\ 1 = 2\ 1,$$

$$f\ 2\ 1 = 1 + 1\ 2 = 3\ 2,$$

$$f\ 3\ 2 = 1 + 2\ 3 = 5\ 3,$$

and so on. Continuing the calculations, we obtain the orbit shown below:

1 1, 2 1, 3 2,5 3,8 5,13 8 ,21 13,34 21 ,55 34,89 55,144 89 ,233 144, 377 233,….

Note that the numerator of each fraction is the denominator of the next fraction. Do you see why this happens?

If we list down the sequence of denominators of these fractions, here is what we obtain:

$$⟨1,1,2,3,5,8,13,21,34,55,89,144,\ldots⟩.$$

This is called the **FIBONACCI SEQUENCE;** it is perhaps the most famous sequence in all of mathematics! Its defining property is that after the first two

terms, each term is the sum of the two terms preceding it. If we denote the nth term of the Fibonacci sequence by Fn, then the sequence may be defined thus: F1 = 1,F2 = 1, and

$$Fn = Fn\text{-}1 + Fn\text{-}2 \text{ for } n > 2.$$

Using this notation the figures given above may be displayed in tabular form thus:

n	1	2	3	4	5	6	7	8	9	10	11	···
F_n	1	1	2	3	5	8	13	21	34	55	89	···

The name 'Fibonacci sequence' honours the Italian mathematician Fibonacci (1170–1250), also known as 'Leonardo of Pisa', who introduced the sequence in a problem concerning the breeding of rabbits. (If we assume that the rabbits breed according to a certain definite rule, then the number of rabbits during each epoch turns out to be a Fibonacci number.) Fibonacci's great achievement was the writing of an extremely influential book called the LIBER ABACI in which he introduced the Hindu-Arabic place value system of numeration to the rest of Europe. Indeed, he described its working in great detail, resulting in the gradual adoption of the system by the community at large. (This turned out to have major consequences.) Fibonacci lived in pre-Renaissance times, and his work was one of the strands that led to the Renaissance.

If you study the Fibonacci sequence very closely, you will soon discover what makes it so special—its patterns are so numerous that they could occupy entire books! There are very few sequences that are so rich in patterns, or which arise in so many different ways in so many varied branches of mathematics. We give below a small sample of these patterns.

A. Let *a, b, c* be any three consecutive Fibonacci numbers; then b2 - ac is equal to ± 1.
   For example, for the triple (5,8,13), with a = 5,b = 8 and c = 13, we get b2 - ac = 82 - 5 · 13 = 84 - 65 = -1.

B. Let a,b,c,d be any four consecutive Fibonacci numbers; then ad - bc is always equal to ± 1.
   For example, for the quadruple (8,13,21,34), with a = 8,b = 13,c = 21 and

d = 34, we get ad - bc = 272 - 273 = -1.

C. Let a,b,c be any two consecutive Fibonacci numbers; then a2 + b2 is itself a Fibonacci number.
For example, from the pair (3,5) we obtain 32 + 52 = 9 + 25 = 34, and from the pair (8,13) we obtain 82 + 132 = 64 + 169 = 233; both 34 and 233 are Fibonacci numbers.

D. Let a,b,c be any three consecutive Fibonacci numbers; then c2 - a2 is itself a Fibonacci number.
For example, from the triple (3, 5, 8) we obtain 82 - 32 = 55, and from (5,8,13) we obtain 1 32 - 52 = 144; both 55 and 144 are Fibonacci numbers.

E. Let a,b,c,d be any four consecutive Fibonacci numbers, and let x = ad, y = 2bc,z = x2 + y2. Then z is an integer—and a Fibonacci number at that. (As a by-product, we obtain a "Pythagorean triple"—i.e., an integer triple (x,y,z) such that x2 + y2 = z2.)
For example, from the quadruple (*2*, 3, *5*, 8) we obtain

$$x = 16, y = 30, z = 256 + 900 = 34;$$

and from (3,5,8,13) we obtain

$$x = 39, y = 80, z = 1521 + 6400 = 89.$$

Observe that both 34 and 89 are Fibonacci numbers.

F. Let a and b be any two Fibonacci numbers; then their gcd is itself a Fibonacci number. Indeed, we find the following remarkable relationship:

$$\gcd(Fm,Fn) = F\gcd(m,n).$$

For example, let m = 12 and n = 18; then gcd(m,n) = 6. Next,

$$F12 = 144 = 24 \times 32, F18 = 2584 = 23 \times 17 \times 19.$$

Therefore, gcd(F12,F18) = 23 = 8. And, indeed, we have F6 = 8.

G. Instead of writing the original sequence as a sequence of fractions, i.e., as

$$1\ 1,2\ 1,3\ 2,5\ 3,8\ 5,13\ 8\ ,21\ 13,34\ 21,55\ 34,\ldots,$$

we write it as a sequence of decimal fractions:

$$\langle 1,2,1.5,1.67,1.6,1.625,1.615,1.619,\ldots\rangle.$$

In this form, it visibly has a limit, and calculations give its value as roughly 1.618. Alongside the Fibonacci sequence, this number too occupies a very prominent place in the "Mathematical Hall of Fame"; it is the *Golden Ratio*, and it has been known since Greek times.

In the Parthenon building, a part of the Acropolis complex built more than two thousand years ago near present-day Athens, the length to height ratio is just this number. The Greeks considered the ratio to be very pleasing to the eye and the best for aesthetic appearances. Interestingly, so did the Renaissance genius Leonardo da Vinci (who was a scientist, engineer, painter and sculptor all in one!); looking through his notebooks one finds rough sketches of various figures (e.g., the human body, the human hand, the form of a horse), each framed by a rectangle of this shape.

The Golden Ratio can be found in 'closed-form'—i.e., expressed in terms of known quantities; it turns out to be equal to 1 2(5 + 1).

H. The orbit listed above,

$$1\ 1,2\ 1,3\ 2,\ 5\ 3,8\ 5,13\ 8\ ,21\ 13,34\ 21,55\ 34,89\ 55,\ldots$$

was obtained from the seed x = 1. What happens if we use other seeds? Clearly x = 2 will give the same limiting ratio. (Why?) Experiment with other seeds: x = 3,x = 4,x = 5….. What do you notice?

You may not find it easy to prove the properties listed above, as the proofs typically need a technique of proof known as "proof by induction:' which you may not know at this stage. Do try, however—you may come up with new proofs! Try also to discover more such properties—there are surely many more waiting to be discovered.

11.6 A route to square roots

In the previous section, we used the function

$$f(x) = 1 + 1\ x = x + 1\ x\ .$$

The function g we use now looks similar, but the results are quite different:

$$g(x) = 1 + 1\,x + 1 = x + 2\,x + 1.$$

Here is the orbit ⟨1;g⟩:

$$1\ 1{,}3\ 2{,}7\ 5{,}17\ 12{,}41\ 29{,}99\ 70{,}239\ 169{,}577\ 408{,}\ldots.$$

As with the Fibonacci sequence, this sequence has many pretty properties hidden within it, all waiting to be discovered! Here are a few for you to examine.

A. Consider the sequence of numerators:

$$\langle 1,3,7,17,41,99,239,\ldots\rangle.$$

If a,b,c are any three consecutive numbers of this sequence, then $a + 2b = c$ and $b2 - ac = \pm 2$.

For example, for the triple (a,b,c) = (17,41,99) we have:

$$a + 2b = 17 + (2 \times 41) = 99 = c,$$

$$b2 - ac = 1681 - 1683 = -2.$$

B. Consider now the sequence of denominators:

$$1,2,5,12,29,70,169,408,\ldots.$$

Amazingly, very similar properties hold for this sequence! That is, if a,b,c are any three consecutive numbers of the sequence, then

$$a + 2b = c \text{ and } b2 - ac = \pm 1.$$

For example, for the triple (12, 29, 70) we have:

$$a + 2b = 12 + (2 \times 29) = 70 = c,$$

$$b2 - ac = 292 - (12 \times 70) = 841 - 840 = 1.$$

C. Next, let a/b be any fraction of the original sequence (a and b are positive integers); then:

$$a2 - 2b2 = \pm 1.$$

For example, for the fractions 17/12 and 41/29,

$$172 - (2 \times 122) = 289 - 288 = +1,$$

$$412 - (2 \times 292) = 1681 - 1682 = -1.$$

D. As in the earlier exercise, we express the terms of the original sequence,

$$1\ 1{,}3\ 2{,}7\ 5{,}17\ 12,\ 41\ 29{,}99\ 70{,}239\ 169{,}577\ 408{,}1393\ 985\ ,\ldots$$

as decimal fractions, and obtain:

$$\langle 1{,}1.5{,}1.4{,}1.42{,}1.414{,}1.4143{,}1.4142\ldots\rangle,$$

in which form it is seen quite visibly to have a limit. You may be astonished to learn that the limit is 2 (approximate value: 1.41421356). Is this not a wonderful discovery?

Try to find out why these properties hold. (In other words, prove them!) You may need to use the "Method of Induction" referred to earlier.

★★★★★★★★★★★★★★★★★★★

You may have noticed the curious commonality of properties of the sequences studied in this and the previous section. What lies behind this commonality? Try to find out its hidden cause. In pursuing the investigation, you may want to investigate other such sequences, for instance, those arising from functions such as:

$$f1(x) = 3x + 1\ x + 1\ ,f2(x) = 5x + 1\ x + 1\ ,f3(x) = x + 3\ x + 1,$$

and so on.

11.7 More on square roots

In this section we discuss two ways of computing square roots. The first one is the traditional 'long division' algorithm (not too well known these days, unfortunately!). The second one uses a curious and elegant relationship between the geometric mean of two numbers and their arithmetic and harmonic means.

The long-division method for computing square roots

We work throughout in base-10. Let the square root of an integer N be

required (N > 0). Suppose that I ≤N < I + 1 for some non-negative integer I. Let k be the first digit in the decimal part of N; then N = I.k…, with more digits to come after the '*k*'. We assume that *I* can be easily found via inspection; we now wish to find k. Once it is found, the digit following '*k*' can be found by working with the number 100N instead of N (for the square root of 100N will be of the form *Ik.m* … for some digit m).

Now the statement that N = I.k… means that

$$I + k\,10 \le N < I + k + 1\,10\,,$$

or (equivalently) that 10I + k ≤ 10N < 10I + (k + 1). By squaring we now get:

$$100I2 + 20Ik + k2 \le N < 100I2 + 20I(k + 1) + (k + 1)2,$$

$$\therefore\ k(20I + k) \le 100(N - I2) < (k + 1)(20I + k + 1).$$

Therefore, k is the largest with the property that k(20I + k) ≤ 100(N - I2). This characterization allows us to find k. Once it is known, the same step can be done again to find the next digit following k, and so on.

Example 8. Say we wish to compute 150. Let N = 150. Then 122 < N < 132, so I = 12 and N - I2 = 6. We must now compute the largest digit k for which k(240 + k) ≤ 600. We easily find that k = 2 (since 2 × 242 = 484 < 600, whereas 3 × 243 > 600). Therefore, 150 = 12.2….

Now we set N = 15000. We have just found out that 1222 ≤ N < 1232; so I = 122. This gives N - I2 = 15000 - 1222 = 116, so 100(N - I2) = 11600. We must now compute the largest digit k for which k(2440 + k) ≤ 11600. We easily find that k = 4; this is so because 4 × 2444 < 1 1600, whereas 5 × 2445 > 11600. So 15000 = 122.4…, ∴150 = 12.24…. We may continue in this manner and find as many decimal digits as desired; we get 150 = 12.2474....

The reader should compute a few square roots by this method, say 200 and 250.

Remark. The reader may wonder whether there is any similar algorithm for computing cube roots; i.e., of the "long-division" type. There is; readers familiar with 'Vedic mathematics' may recall studying such algorithms. However, they are much more complicated to work through than the algorithm for square roots.

Square roots via the arithmetic and harmonic means

Another curious iterative method for the computation of square roots is the following.

Given two positive numbers a and b, we associate three kinds of means with them—the arithmetic mean (AM), geometric mean (GM) and harmonic mean (HM), defined as follows:

$$AM = a + b\ 2\ ,\ GM = ab,\ HM = 2ab\ a + b.$$

Note that the HM of a list of numbers is the reciprocal of the AM of their reciprocals.

So if a = 1 and b = 9 then AM = 5, GM = 3, HM = 1.8. .Observe that HM < GM < AM. It tums out that if a and b are unequal, then these inequalities always hold. (If a = b then all three means coincide.) It is easy to show algebraically why this is so. The inequality HM < GM, i.e.,

$$2ab\ a + b < ab,$$

can be written after some simplification as

$$a + b\ 2 > ab, \quad \text{or} \quad AM > GM,$$

and this may in tum be written, after squaring and simplifying, as

$$a2 + 2ab + b2 > 4ab, \quad \text{or} \quad a2 - 2ab + b2 > 0.$$

But this is clearly true, as a2 - 2ab + b2 = (a - b)2, and a non-zero square is always positive.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

The computations of the AM and HM involve only **RATIONAL OPERATIONS** (addition, subtraction, multiplication and division), whereas computation of the GM involves a square root (considered to be a non-rational operation). Rather curiously, we may zero in on the GM by computing only AMs and HMs, as follows. Given two positive numbers a and b, say with a < b, we define the pairs (a0,b0),(a1,b1),(a2,b2),… thus: a0 = a,b0 = b, and for all i ≥ 0,ai+1 = HM of ai and bi, and bi+1 = AM of ai and bi. Note that the scheme is iterative in nature; at each step, we use only the preceding pair of numbers.

It will be found that the gap between ai and bi steadily decreases, and at quite a rapid rate too; and "in the limit" both ai and bi tend to the common value ab, i.e., to the GM.

Example 9. Let a=1 and b = 9. The computations are displayed as follows.

i	a_i	b_i
1	1.8	5.0
2	2.647058824	3.4
3	2.976653696	3.023529412
4	2.999908449	3.000091554
5	2.999999999	3.000000001

Observe that a5 and b5 are practically the same.

★ ★

To prove that ai and bi get steadily closer to ab as i gets larger and larger, we argue as follows. Since

$$a1 = 2ab\ a + b, b1 = a + b\ 2\ ,$$

we get a1 × b1 = a × b. Since a2,b2 are defined in terms of a1,b1 in the same way as a1,b1 are defined in terms of a,b we similarly get a2 × b2 = a1 × b1. Extending this result in the obvious way, we see that

$$ai \times bi = a \times b \quad \text{for all n.}$$

So the product ai × bi is an INVARIANT; it stays the same all through the iteration.

Now consider the difference b1 - a1; we have

$$\begin{aligned} b1 - a1 &= a + b\ 2 - 2ab\ a + b \\ &= (a + b)2 - 4ab\ 2(a + b) \\ &= b - a\ 2 \cdot b - a\ b + a. \end{aligned}$$

Since (b - a)/(b + a) is non-negative and less than 1, we get:

$$0 \le b1 - a1 < b - a\ 2\ .$$

Similarly we get 0 ≤ b2 - a2 < (b1 - a1)/2, and therefore

$$0 \le b2 - a2 < b - a\ 4 .$$

In general we have $0 \le bi - ai < (b - a)/2i$. Since 2i grows without bound as i increases, we conclude that "in the limit", ai and bi become equal; and as their product is ab, it follows that each of the limiting values is ab, i.e., the GM.

A graphical approach

The argument followed above has a nice graphical interpretation. Let n = ab, and consider the curve whose equation is xy = n, i.e., y = n/x (see Figure 11.1). Of relevance to us is the portion of the curve which lies in the first quadrant. Note that the point P0 with coordinates (a,b) lies on the curve. Let Q0 be the point with coordinates (b,b); then Q0 lies to the *right* of P since b > a.

Let P0Q1 be the normal from P0 to the line y = x; the foot of the normal has coordinates

$$a + b\ 2\ ,a + b\ 2\ .$$

Let the horizontal line through Q1 meet the curve at P1. Then P1 has coordinates

$$2ab\ a + b'a + b\ 2\ .$$

The movement from P0 *to* P1 *is taken to be the first round of the iteration*. We now continue the iteration and get a sequence of points P0,P1,P2,… on the curve, and Q0,Q1,Q2,… on the line y = x.

Let the line Q1P1 be extended beyond P1 to R, where it meets the line x = a; then R has coordinates (a,(a + b)/2). It is clear via similar triangles that RQ1 = P0Q0/2; also, P1Q1 < RQ1; therefore we get

$$P1Q1 < P0Q0\ 2\ .$$

Similarly we get P2Q2 < P1Q1/2, and so P2Q2 < P0Q0/4; and so on. So the gap between Pi and Qi decreases rapidly, and "in the limit" Pi and Qi coincide. The point of coincidence is the point where the curve xy = n and the line y = x meet in the first quadrant—the point (n,n).

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

So to compute n for some given positive number n, we compute the pairs

(a0,b0),(a1,b1),(a2,b2),…, where

a_0= l, b_0= n; and for $i \geq 0$,

a_{i+1} = HM of $\{a_i, b_i\}$, b_{i+1}= AM of $\{a_i, b_i\}$.

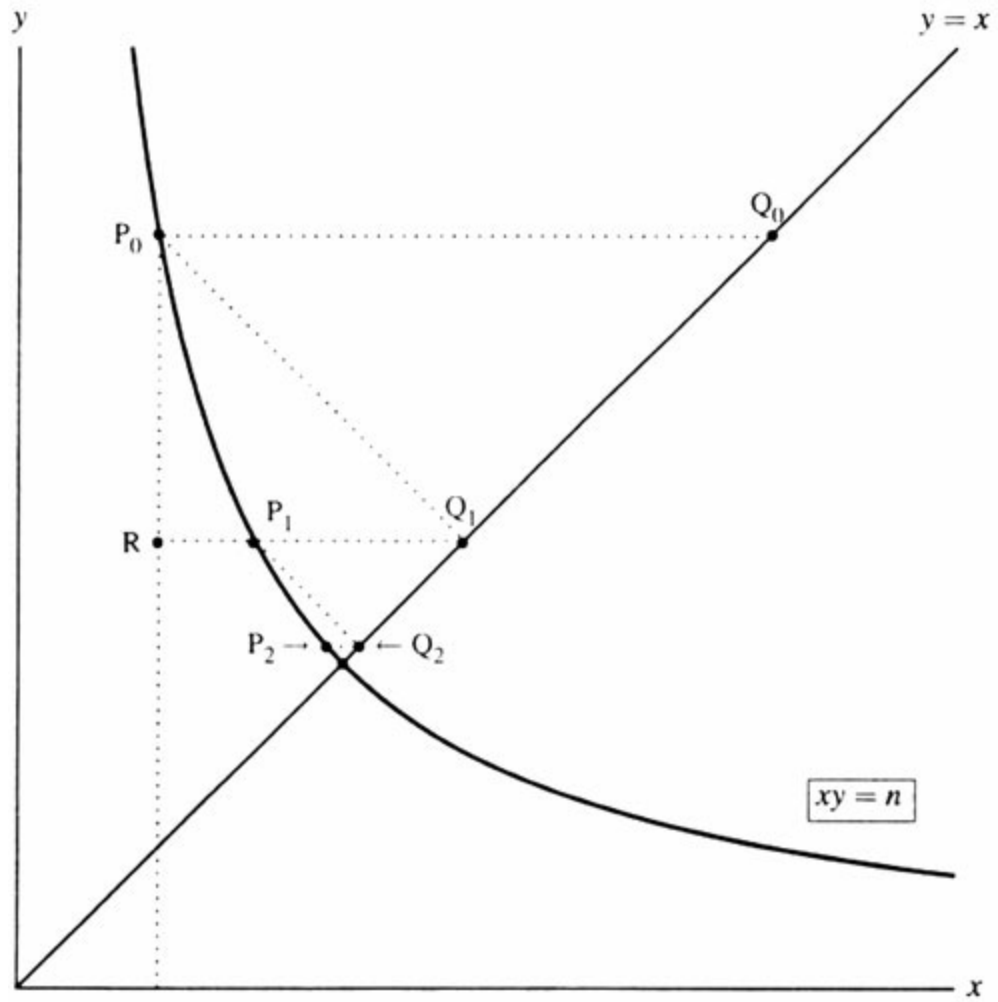


Figure 11.1. *Square roots via the arithmetic mean and harmonic mean*

A few iterations will suffice to yield a fairly accurate value of n.

The arithmetic-geometric mean

We have just seen how the GM of two positive numbers a and b may be estimated via repeated AM-HM computations. So we might say that the geometric mean is an *arithmetic-harmonic mean*, or GM = AHM.

What happens if, instead, we do repeated computations of AM and GM? In

other words, given two positive numbers a and b, if we define a0 = a,b0 = b, and for i ≥ 0,

$$ai+1 = GM \text{ of } \{ai,bi\}, bi+1 = AM \text{ of } \{ai,bj\},$$

what can be said about the pairs (ai,bi) as i increases? Computationally we find that the gap between ai and bi decreases at a very rapid rate. For instance here is the result when a = 1 and b = 9:

i	a_i	b_i
0	1.0	9.0
1	3.0	5.0
2	3.872983346	4.0
3	3.935979343	3.936491673
4	3.936235499	3.936235508
5	3.936235504	3.936235504

It is quite easy to show that as i grows without limit, the numbers ai and bi approach some common limiting value. This number is the arithmetic - geometric *mean* (AGM) of a and b; we write it as AGM (a,b). So AGM (1, 9) is approximately 3.936.

In the case of the arithmetic-harmonic mean we obtained a very simple and neat result: *the arithmetic-harmonic mean is the same as the geometric mean*. But no such simple result holds for the AGM. On the contrary, the following most surprising equality holds (a knowledge of calculus is needed to even understand the statement!): if a,b > 0 then

$$AGM\ (a,b) = \pi 2 \int 0\pi/2 (a2\cos 2\theta + b2\sin 2\theta)\text{-}1/2 d\theta\ .$$

The result is due to the "prince of mathematicians", Karl Friedrich Gauss. There is, unfortunately, no question of proving it here. The reader is referred to B Sury's article, *The Arithmetico-Geometric Mean of Gauss*, in the journal RESONANCE (August 2000).

11.8 Cube roots

The ideas developed in the preceding section may be extended quite easily to the case of cube roots. The function governing the iteration needs only a small modification. Thus, given two positive numbers a and b, we define the

pairs (ai,bi) for i ≥ 0 thus: a0 = a,b0 = b, and for i ≥ 0,

$$ai+1 = ai + bi\ 2\ ,bi+1 = .\ ab\ ((ai + bi)/2)2.$$

Example 10. Let a = 1 and b = 8; the pairs (ai,bi) then take the following values.

i	a_i	b_i
0	1.0	8.0
1	4.5	0.3950617284
2	2.447530864	1.335468451
3	1.891499658	2.236029109
4	2.063764383	1.87832077
5	1.971042577	2.059197374
6	2.015119975	1.970099545
7	1.99260976	2.014862808
8	2.003736284	1.992548319
9	1.998142302	2.00372058
10	2.000931441	1.998138419

Observe that as i increases, ai and bi both approach 2. However the rate at which the common value of 2 is approached is not as impressive as in the case of square roots. In the general case, as i increases without limit, ai and bi both approach the cube root of a2b.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

As earlier, this iteration admits of a nice graphical interpretation; see Figure 11.2. Let a2b = n. We consider the curve y = n⁄x2; then the point P0 = (a,b) lies on the curve. We now locate points Q0,P1,Q1,P2,…, in that order, as follows: Q0 is the foot of the normal from P0 to the line y = x;P1 is the point on the curve 'vertically' below Q0 (so its x-coordinate is the same as that of Q0) ;Q1 is the foot of the normal from P1 to the line y = x;P2 is the point on the curve vertically below Q1;…. It will be seen (and to prove this is easy) that the points P1,P2,P3, ... approach the point of intersection of the line y = x and the curve y = n⁄x2, i.e., the point with coordinates (n1⁄3,n1⁄3).

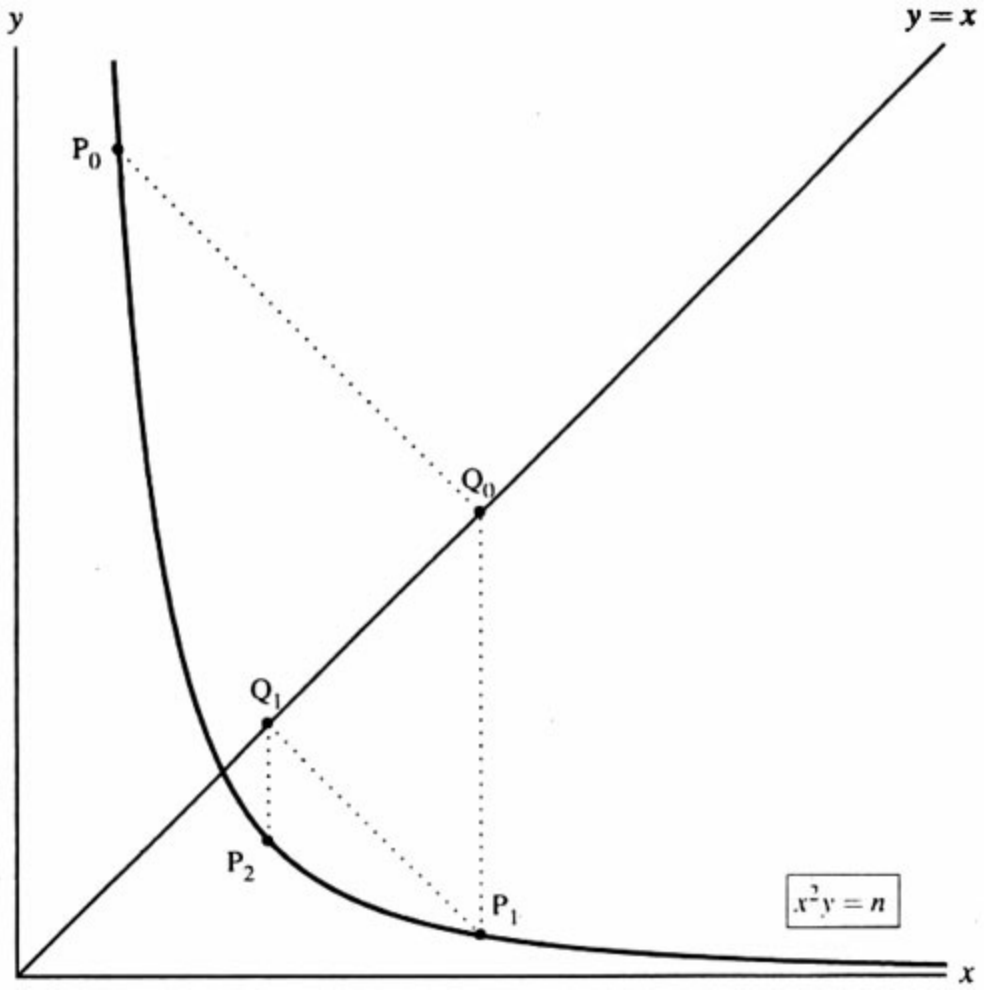


Figure 11.2. *Cube roots via iteration*

Extension to higher roots

Having seen how cube roots can be computed, the extension to higher roots is immediate. Let k ≥ 2 be a fixed integer. Given two positive numbers a and b, we compute the pairs (ai,bi) for i ≥ 0 thus: a0 = a,b0 = b, and for i ≥ 0,

$$a_{i+1} = a_i + b_i\ 2\ ,b_{i+1} = a_{k-1}b\ ((a_i + b_i)/2)_{k-1}.$$

As i grows without bound, the numbers ai and bi begin to approach a common limit whose value is

$$a_{k-ı}b_{1/k}$$

The graphical interpretation is clear: we work on the curve y = n/xk-1 where n = ak-1b. As the iteration progresses, we approach the point of intersection of

the curve and the line y = x.

So by setting a = 1 and b = n we can estimate the kth root of n. Note however that for large values of k, the speed at which the values converge to the answer is not too impressive.

Remark. We shall see in Volume II that the procedures described above are special cases of the *Newton-Raphson method*, also known as *Newton's method, a*s applied to the computation of square roots and cube roots. This is an algorithm of great utility in solving equations numerically (as different from solving them in "closed form").

Chapter 12

A Miscellany–II

In this chapter we meet another mixed bag of iterations.

12.1 SP numbers

In the March 1999 issue of THE MATHEMATICAL GAZETTE, in an interesting article titled "SP Numbers", the author defines a positive integer n to be a SP number if it equals the product of the sum of its digits and the product of its digits (with the number written in base-10). For example, 1 is trivially a SP number,and so are 135 and 144, because:

$$135 = (1 + 3 + 5) \times (1 \times 3 \times 5),$$

$$144 = (1 + 4 + 4) \times (1 \times 4 \times 4).$$

Are there any other SP numbers? The author shows that these are (surprise!) the only SP numbers. A companion article in the same issue establishes the same conclusion in a different manner.

Our purpose here will be study the problem through some of the machinery developed earlier. We define a function SP which acts on positive integers via their base-10 representation, as follows:

$$\text{SP (n)} = \text{sum of digits of n} \times \text{product of digits of n.}$$

The problem then is to find the fixed points of SP. We may ask, more generally, for the cycle structure of SP.

We first show that if n has more than 85 digits, then SP (n) < n; that is,

$$\text{if } n > 1084, \text{ then SP (n)} < n.$$

Proof. Denote by k the number of digits that *n* has; then

$$10k\text{-}1 \leq n < 10k.$$

Now note that (a) the sum of the digits of n cannot exceed 9k, and (b) the product of the digits of n cannot exceed 9k. (These limits are attained only when n is of the form 999... 99.) Therefore:

$$SP\ (n) \leq 9k \times 9k.$$

We shall now show that if $k \geq 85$, then $9k \times 9k < 10k\text{-}1$. With this established, it will follow that if $k \geq 85$, then $SP\ (n) < 10k\text{-}1$ and therefore that $SP\ (n) < n$.

Claim. If k ≥ 85then-9k × 9k < 10k-1.

Proof. The inequality to be proved may be written in the form 90k × 9k < 10k, which we rewrite as (10⁄9)k > 90k, So we must establish the following:

$$\text{If } k \geq 85, \text{ then } 1\ k\ 10\ 9\ k > 90.$$

Let the function f be defined on N by.

$$f(k) = 1\ k\ 10\ 9\ k$$

Then f(1) ≈ 1.111,f(2) ≈ 0.6173,f(3) ≈ 0.4572, and so on. Now consider the quantity f (k + 1)⁄f(k). We get, after some computation:

$$f(k + 1)\ f(k) = k\ k + 1 \times 10\ 9 = 10k\ 9k + 9.$$

For the quantity 10k⁄(9k + 9) to be more than 1 we must have 10k > 9k + 9, or k > 9. So if k > 9 then f(k + 1) > f(k). In other words, *the sequence*

$$f(10), f(11), f(l2), f(13), f(14), \ldots$$

steadily increases.

Using the computer we now construct the following table of data.

k	20	30	40	50	60	70	80	85
$f(k)$	0.411	0.786	1.69	3.88	9.27	22.8	57.2	91.2

The 'strict increase' can be visibly seen in the table. Observe that f(85) > 90. It follows that if k > 85, then f(k) > 90. In other word, f(k) *greater than 90 for* k ≥ 85, *and therefore* 9k × 9k < 10k-1.fork ≥ 85. This implies, as pointed out earlier, that if n > 1084, then SP (n) < n.

The implication of this is that all SP-fixed points can be found among the integers below 1084. This may not seem too helpful a result!—but it does provide a starting point.

Remark. The upper bound of 1084 can be brought down to 1060 without too much effort.

★ ★

To make progress, we allow the function SP to act on sets (and not merely on single numbers), and define a function g which acts as follows: if X is any set of numbers, then

$$g(X) = SP(X) \cap X.$$

Example 1. Let X = {11,12,13,14,15}. Then

$$SP(X) = \{2,6,12,20,30\}, g(X) = \{\};$$

so g(X) = ∅, the empty set.

If we iterate the function g on the set

$$1,2,3,\ldots,1084 - 1,$$

then ultimately we will obtain *all* the fixed points and *all* the cycles of SP. Unfortunately the set is much too large, and even with a powerful computer we will not make appreciable progress.

The result of iterating g on the set S, where

$$S = \{1,2,3,\ldots,197,198,199\},$$

is shown in Table 12.1. For convenience we write g(m) to mean that g is applied m times in a row; thus

$$g(2)(X) = g(g(X)), g(3)(X) = g(g(g(X))),$$

and so on (g(1) is the same as g).

Since g(8)(S) = g(7)(S), it follows that {1, 135, 144} is an SP-invariant set, implying that each of its elements is part of a cycle.This agrees with what we already know; indeed, 1, 135 and 144 are fixed points.

Completing the search

The analysis at this stage gets a bit complicated, and we shall only quote some results from the two articles referred to above, where various conditions are shown to hold for any SP number. Specifically, denoting the sum of the digits of n by S, the product of the digits by P, and the number of digits of n by k, the following are established.

k	$g^{(k)}(\{1,2,3,\ldots,197,198,199\})$
1	{1, 2, 3, 4, 6, 8, 9, 12, 15, 16, 20, 24, 25, 30, 35, 36, 42, 48, 49, 54, 56, 63, 64, 70, 72, 80, 81, 84, 90, 96, 99, 108, 120, 126, 128, 135, 140, 144, 160, 162, 176, 180, 198}
2	{1, 4, 6, 9, 16, 30, 36, 42, 48, 64, 70, 72, 81, 108, 120, 126, 135, 144, 162, 176, 180}
3	{1, 16, 36, 42, 48, 72, 81, 108, 126, 135, 144, 162}
4	{1, 42, 48, 72, 108, 126, 135, 144, 162}
5	{1, 48, 108, 126, 135, 144}
6	{1, 108, 135, 144}
7	{1, 135, 144}
8	{1, 135, 144}

Table 12.1. *Result of iterating on* {1,2,…,198,199}

a. Since no digit of n is 0, and the prime numbers below 10 are 2, 3, 5 and 7, p must be of the form $2a \times 3b \times 7d$ or $3b \times 5 \times 7d$.

b. If 3 is a divisor of P. then 3 is a divisor of n too, ∴ 3 is a divisor of S, making 9 a divisor of n and therefore of S as well.

c. If 3 is a divisor of P, then $S \geq 9k - 27$ (the proof of this is quite involved, but the result obviously simplifies the search very greatly).

d. From the preceding results we see that there are only six cases to consider: (i) $S = 9k$ (ii) $S = 9k - 9$ (iii) $S = 9k - 18$ (iv) $S = 9k - 27$ (v) $P = 2a \times 7d$ (vi) $P = 5c \times 7d$.
It is shown that there are no SP numbers with $S = 9k$; at most 1056 SP numbers with $S = 9k - 9$; at most 6480 SP numbers with $S = 9k - 18$; at

most 16786 SP numbers with S = 9k - 27; at most 4848 SP numbers with P = 2a × 7d; and precisely one SP number with P = 5c × 7d, namely n = 1.

The potential candidates are now small in number. and a computer search becomes feasible (the number of candidates to be tested is of the order of a hundred thousand. and nothing like 1060, which would make the task hopeless). After a relatively short length of time the computer produces a list of all SP numbers. The list turns out to be rather small (perhaps disappointingly so!)—it has just the numbers 1,135 and 144.

k	$b^{(k)}(\{1,2,3,\ldots,98,99,100\})$
1	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18. 19, 20, 23, 24, 26, 27, 29, 31, 34, 35, 39, 41, 44, 47, 48, 49, 53, 55, 59, 62, 63, 69, 71, 79, 80, 89, 99}
2	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 23, 24, 26, 27, 29, 34, 35, 39, 41, 44, 49, 59, 69, 79, 89, 99}
3	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 23, 24, 29, 39, 49, 59, 69, 79, 89, 99}
4	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 29, 39, 49, 59, 69, 79, 89, 99}
5	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 29, 39, 49, 59, 69, 79, 89, 99}

Table 12.2. *Result of iterating* b *on* {1,2,3,…,99,100}

Remark. Unfortunately, this analysis reveals nothing about any higher order cycles of SP (even whether they exist).

What about S+P numbers?

Much the same analysis can be conducted for what we may call the "S + P" function. Write S(n) for the sum of the digits, and P(n) for the product of the digits of n (with n written in base-10), and let a(n) = S(n) + P(n). What can be

said about the cycle structure of a?

Define a function *b* which acts on sets of numbers thus: $b(X) = a(X) \cap X$ (we allow a too to act on sets of numbers). If we iterate b on the set S of integers below 100, we obtain Table 12.2.

Observe that $b(4)(S) = b(5)(S)$. This means that something interesting may be found in the *a*-invariant set T, where

$$T = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,$$

$$17, 18, 19, 29, 39, 49, 59, 69, 79, 89, 99\}.$$

By computing the *a*-value of each number in T, we find that $n = a(n)$ (i.e., $n = S(n) + P(n)$) for

$$n \in \{19,29,39,49,59,69,79,89,99\}.$$

This, then, is the list of fixed points of a. Amazingly, we find that the other elements in T are pm of a single cycle! So we obtain an 18-cycle:

$$1,2,4,8,16,13,7,14,9,18,17,15,11,3,6,12,5,10.$$

Whew!

Curiously, the cycle displayed above is *identical* to the cycle of remainders of the powers of 2 (mod 19). That is, if we compute the remainders when we divide the numbers

$$20,21,22,23,...,215,216,217$$

by 19, we obtain exactly the same numbers as those listed above, and in the very same order. A pretty coincidence?—not quite! The reader is invited to inquire further.

And what about S-squared plus P-squared numbers?

We could go on in this vein in and ask whether there are any numbers n such that

$$n = S(n)2 + P(n)2.$$

It appears that there is only one such integer, namely 41. But we leave the analysis to the reader. (The author's computer search revealed precisely one

fixed point and no higher-order cycles.)

12.2 The SSQ iteration revisited

In Chapter 5 we studied the SSQ iteration,

$$n \mapsto \text{sum of the squares of the base ten digits of } n.$$

Here is more material about this iteration, taken from an interesting article by Alan Beardon in the November 1998 issue of THE MATHEMATICAL GAZETTE. The author generalizes the function to cover different number bases by defining the function FB to act on N as follows: if the number n has the base-*B* expansion

$$n = a0 + a1B + a2B2 + \cdots + a\,rBr,$$

where the aj (the digits of n in base B) satisfy the condition $0 \geq aj \leq B - 1$ and $ar > 0$, then

$$FB(n) = a02 + a\,12 + a\,22 + \cdots + a\,r2.$$

Thus SSQ is identical to F10.

Happy numbers and sad numbers

Observe that 1 is a fixed point for each FB. The author defines a number n to be *happy* if by iterating FB on n we ultimately reach 1; i.e., if FB(k)(n) = 1 for some number k. If it gets trapped in a cycle of length greater than 1, then n is said to be *sad*.

For the case B = 10 we had found earlier that there is just one fixed point (1), and that every number is either happy or sad. What can be said for other bases? Is it true that every number is either happy or sad no matter what base we choose? For which base does it happen that 1 is the sole fixed point? Is there a base for which there are precisely two fixed points? (The answer: no.) The author uncovers a great deal of curious facts of this kind.

An easy result for base 2

The author prefaces his analysis with the following easy result: *In base 2, every number is happy*. This holds because in base 2, each aj is 0 or 1, resulting in

the pleasing identity aj2 = aj for all j. Now consider a number n > 1. Then r ≥ 1, so 2r > 1, and therefore

$$\begin{aligned} F2(n) &= a02 + a\,12 + a\,22 + \cdots + a\,r2 \\ &= a0 + a1 + a2 + \cdots + ar \\ &< a0 + a1 \cdot 2 + a2 \cdot 22 + \cdots + a\,r \cdot 2r = n. \end{aligned}$$

So F2(n) < n for all n > 1, implying that for n > 1, the sequence

$$n, F2(n), F2(F2(n)), F2(F2(F2(n))), \ldots$$

decreases *strictly* till 1 is reached (which must necessarily happen), after which no more changes can happen. So every n reaches 1 under repeated iteration.

Some isolated facts

Here are a few isolated facts concerning the maps FB for B = 3,4,5,7 and 20.

- *The case* B = 3. The map F3 has three fixed points (1, 5 and 8), and one 2-cycle (⟨2,4⟩).
  Verification. To check that 5 is a fixed point, observe that 5 = (12)3 and that 12 + 22 = 5; and to check that ⟨2,4⟩ is a 2-cycle, observe that 2 = (2)3 and 4 = (11)3, therefore we get

  $$2 \mapsto 22 = 4 \mapsto 12 + 12 = 2.$$

- *The case* B = 4. The map F4 has one fixed point 1, and no cycles.
- *The case* B = 5. The map F5 has three fixed points (1, 13 and 18), and one 3-cycle (⟨4,16,10⟩).
- *Verification*. 13 = (23)5, so F5(13) = 22 + 32 = 13. For the 3-cycle, observe that 4 = (4)5,16 = (31)5 and 10 = (20)5, and that 42 = 16,32 + 12 = 10,22 = 4.
- *The case* B = 7. The map F7 has five fixed points (1, 10, 25, 32 and 45), and two 4-cycles (⟨2,4,16,8⟩ and ⟨17,13,37,29⟩).

  The verification is left to the reader.
- *The case* B = 20. The map F20 has just one fixed point (1), and one 26-

cycle.

The author now proceeds to make some sense of these facts, and to find a general approach that knits them together.

Various claims

Let n be, in base B, a number with r digits, i.e.,

$$n = a0 + a1B + a2B2 + \cdots + a\ r\text{-}1Br\text{-}1$$

with $0 \le aj \le B - 1$ and $ar\text{-}1 > 0$. The author proceeds to establish the following results concerning the fixed points of FB. In each statement n refers to a positive number written in base B.

Claim 1. *Suppose that* n *has at least four digits*; *then* FB(n) *has fewer digits than* n. *In other words, if* $n \ge B3$, *then* $FB(n) < n$.

Claim 2. *If* n *has at most three digits, then so does* FB(n).

Claim 3. *The cycles of* FB *all consist of numbers with at most three digits.*

In other words, the following set contains all the cycles of FB:

$$\{1,2,3,\cdots,B3 - 1\}$$

This result may be substantially improved, yielding Claim 4.

Claim 4. *The following set contains all the cycles of* FB:

$$\{1,2,3,...,2B2 - 1\}.$$

When applied to the case B = 10 (corresponding to our original SSQ iteration), the result implies that the cycles of SSQ all lie in the set {1, 2,3, …,199 } . And indeed this is so: there is just one fixed point 1, and one 8-cycle

$$\langle 4,16,37,58,89,145,42,20 \rangle,$$

whose entries all lie below 199.

Claim 5. *If* n *is a fixed point of* FB *and* $n > 1$, *then* n *is a two digit number*; *that is*,

$$n \in \{B,B + 1,B + 2,...,B2 - 1\}.$$

This result quickly yields a practical method for finding all the fixed points of FB: we simply have to consider all n of the form a + bB with a,b ∈{0,1,2,…,B - 1}, and check whether or not a2 + b2 = a + bB.

★ ★

In fact the result implies a great deal more. The relation a2 + b2 = a + bB may be written as

$$4a2 - 4a + 4b2 - 4bB = 0,$$

and then as

$$2a - 12 + 2b - B2 = 1 + B2.$$

This means that each fixed point of FB *yields an expression for* 1 + B2 *as a sum of two squares, and conversely*. What a surprising revelation!

For example, consider B = 10 (this corresponds to the original SSQ iteration). Then 1 + B2 = 101, and there is just one way of writing 101 as a sum of two squares:

$$101 = 1 + J00 = 12 + 102.$$

Allowing for signs, we get:

$$101 = (\pm 1)2 + (\pm 10)2.$$

Examining the equation (2a - 1)2 + (2b - B)2 = 1 + B2, we arrive at the following possibilities:

$$2a - 1 = \pm 1, 2b - 10 = \pm 10,$$

$$2a - 1 = \pm 10. 2b - 10 = \pm 1.$$

The second possibility does not yield anything (2*a* - 1 must be an odd number, and so cannot equal ± 10). Since *a.b* ∈{0.1,2,…,9}, the first possibility yields *a* = 0 or 1, and *b* = 0. This corresponds to the fixed points 0 and 1. (Generally do not list 0 as a fixed point, because 0∈N.)

Or consider the case B = 7, with 1 + B2 = 50. There are *two* ways of writing 50 as a sum of two squares:

$$50 = 12 + 72 = 52 + 52.$$

Allowing for signs, we get the following representations:

$$50 = (\pm 1)2 + (\pm 7)2 = (\pm 5)2 + (\pm 5)2.$$

Examining the equation $(2a - 1)2 + (2b - B)2 = 1 + B2$, we arrive at the following possibilities:

$$2a - 1 = \pm 7, 2b - 7 = \pm 1;$$

$$2a - 1 = \pm 1, 2b - 7 = \pm 7;$$

$$2a - 1 = \pm 5, 2b - 7 = \pm 5.$$

Solving these equations, and keeping in mind the restriction $a,b \in \{0,1, \ldots,5,6\}$, we obtain the five fixed points of F7, namely: 1, 10, 25, 32 and 45.

Claim 6. *If* $1 + B2$ *is prime, then the map* FB *has just one fixed point. Conversely, if* FB *has just one fixed point, then* $1 + B2$ *is prime.*

This happens when $B = 2,4,6$ and 10; and for many more values of B. In each of these instances, the only fixed point of FB is 1.

Remark. It is conjectured that there are infinitely many values of B for which $1 + B2$ is prime; but no proof has yet been found! Here is a list of values of B below 100 for which $1 + B2$ is prime:

$$2,4,6,10,14,16,20,24,26,36,40,54,56,66,74,84,90,94.$$

Claim 7. *The number of fixed points* FB *is odd.*

In fact much more can be said; for example:

- the number of fixed points of FB is 3 precisely when $1 + B2$ is a product of two distinct primes (i.e., B is of the form $p \times q$ for distinct primes p and q; this happens, for example, when $B = 3$ or $B = 5$);
- the number of fixed points of FB is 5 precisely when $1 + B2$ is of the form $p \times q2$ for distinct primes p and q (this happens, for example, when $B = 7$; then we get $1 + B2 = 50 = 2 \times 52$).

The reader is urged to read the original article for the proofs of these interesting results.

12.3 Averages

The iteration discussed here may vaguely remind you of the four-number iteration. It too is done on **TUPLES** of numbers: triples, quadruples, 5-tuples, 6-tuples, etc.; more generally, on n-tuples. The iteration involves nothing more complicated than computing averages, but the results hold some pretty surprises.

For ease of description, we describe what to do in the case of a quadruple. Say we are given a starting quadruple (a,x0,y0,b). As the iteration proceeds, the first and last members of the quadruple stay the same, whereas the second and third change repeatedly. Write (a,xn,yn,b) for the quadruple at the nth stage; then we have the following rule:

$$xn+1 = a + yn\ 2\ , yn+1 = xn + b\ 2\ .$$

That is, xn+1 is the simple arithmetic average of a and yn, and yn+1 is the simple arithmetic average of b and xn. The iteration may therefore be written as follows:

$$(a,xn,yn,b) \mapsto a,a + yn\ 2\ ,\ xn + b\ 2\ ,d.$$

We do the averaging repeatedly; what happens?

The above description referred to quadruples. For 5-tuples, we have

$$(a,xn,yn,zn,b) \mapsto a,a + yn\ 2\ ,\ xn + zn\ 2\ ,\ yn + b\ 2\ ,b,$$

and similarly for higher order tuples. The rule is: *the first and last entries always stay the same*; *for the other entries, replace each one by the simple arithmetic average of its immediate neighbours.*

Examples

A few examples will show how the averaging works.

Example 2. We work with triples here. Let the starting triple be (2, 1, 10); we get:

$(2,1,10) \mapsto (2,6,10) \mapsto (2,6,10) \mapsto (2,6,10) \mapsto \cdots$.

We have zeroed in on a fixed point in a single step! It is easy to see that this

will happen regardless of which triple we start with. Indeed we have:

$$(a,x0,b)\mapsto(a,a+b\ 2\ ,b)\mapsto(a,a+b\ 2\ ,b)\mapsto\cdots.$$

The choice of x0 here has no relevance at all. Note that any triple of the form (a,(a + b)⁄2,b) is a fixed point; so there are infinitely many fixed points.

This example has not yielded any surprises, but if we increase the number of entries, then things become a lot more interesting.

Example 3. Now we work with quadruples. Starting with (2,1,7,11), we get the following sequence of quadruples (read it from left to right, top to bottom):

(2, 1, 7, 11),
(2.0, 4.0, 7.75, 11.0),
(2.0, 4.7 5, 7.937, 11.0),
(2.0, 4.937, 7.984, 11.0),
(2.0, 4.984, 7.996, 11.0),
(2.0, 4.996, 7.999, 11.0),
(2.0, 4.999, 8.0, 11.0),
(2.0, 5.0, 9.0, 11.0),
(2.0, 4.5, 6.0, 11.0),
(2.0, 4.875, 7.5, 11.0),
(2.0, 4.969, 7.975, 11.0),
(2.0, 4.992, 7.969, 11.0),
(2,0, 4.998, 7.992, 11.0),
(2.0, 5.0, 7.999, 11.0),
(2.0, 5.0, 9.0, 11.0),
…

We have reached a fixed point: the quadruple (2, 5, 8, 11).

Example 4. The starting quadruple is (3,0,5,24). We get:

(3, 0, 5, 24),
(3.0, 7.5, 14.0, 24.0),
(3.0, 9.375, 16.25, 24.0),
(3.0, 9.844, 16.81, 24.0),
(3.0, 9.961, 16.95, 24.0),
(3.0, 9.99, 16.99, 24.0),
(3.0, 9.998, 17 .0, 24.0),
(3.0, 9.999, 17 .0, 24.0),
(3.0, 10.0, 17.0, 24.0),
(3.0, 4.0, 12.0, 24.0),
(3.0, 8.5, 15.75, 24.0),
(3.0, 9.625, 16.69, 24.0),
(3.0, 9.906, 16.92, 24,0),
(3.0, 9.977, 16.98, 24.0),
(3.0, 9.994, 17.0, 24.0),
(3.0, 9.999, 17.0, 24.0),
(3.0, 10.0, 17.0, 24.0),
…

We have reached the fixed point (3, 10, 17, 24).

Example 5. The starting quadruple is (0,0,0,75). We get:

(0, 0, 0, 75),
(0, 18.75, 37.5, 75.0),
(0, 23.4, 46.97, 75.0),
(0, 24.61, 49.22, 75.0),
(0, 24.9, 49.9, 75.0),
(0, 24.98, 49.95, 75.0),
(0, 24.99, 49.99, 75.0),
(0, 25.0, 50.0, 75.0),
(0, 25.0, 50.0, 75.0),
(0, 0, 37.5, 75.0),
(0, 18.75, 46.97, 75.0),
(0, 23.4, 49.22, 75.0),
(0, 24.61, 49.9, 75.0),
(0, 24.9, 49.95, 75.0),
(0, 24.98, 49.99, 75.0),
(0, 24.99, 50.0, 75.0),
(0, 25.0, 50.0, 75.0),
…

We have reached the fixed point (0,25,50,75).

Remark. In each of the above examples there is convergence to a quadruple whose entries increase by fixed amounts from the first number to the last number; i.e., a quadruple whose entries form an A.P. (arithmetic progression). Does this always happen?

Example 6. We now work with 5-tuples.with (1,1,11,7,21) as the seed, we get:

(1, 1, 11, 7, 21),
(1.0, 2.5, 11.0, 12.5, 21.0),
(1.0, 4.25, 11.0, 14.25, 2 1.0),
(1.0, 5.125, 11.0, 15. 12, 21.0),
(1.0, 5.562, I I .0, 15.56, 21 .0) ,
(1.0, 5.781, 11.0, 15.78, 21.0),
(1.0, 5.891, 11.0, 15.89, 21.0),
(1.0, 5.945, 11.0, 15.95, 21.0),
(1.0, 5.973, 11.0, 15.97, 21.0),
(1.0, 5.986, 11.0, 15.99, 21.0),
(1.0, 5.993, 11.0, 15.99, 21.0),
(1.0, 6.0, 4.0, 16.0, 21.0),
(1.0, 6.0, 7 .5, 16.0, 21.0),
(1.0, 6.0, 9.25, 16.0,21.0),
(1.0, 6.0, 10.12, 16.0, 21.0),
(1.0, 6.0, 10.56, 16.0, 21.0),
(1.0, 6.0, 10.78, 16.0, 21.0),
(1.0, 6.0, 10.89, 16.0, 21.0),
(1.0, 6.0, 10.95, 16.0, 21.0),
(1.0, 6.0, 10.97, 16.0, 21.0),
(1.0, 6.0, 10.99, 16.0, 21.0),
(1.0, 6.0, 10.99, 16.0, 21.0),

(1.0, 5.997, 11.0, 16.0, 21.0),
(1.0, 5.998, 11.0, 16.0, 21.0),
(1.0, 5.999, 11.0, 16.0, 21.0),
(1.0, 6.0, 11.0, 16.0, 21.0),
(1.0, 6.0, 11.0, 16.0, 21.0),
(1.0, 6.0, 11.0, 16.0, 21.0),
(1.0, 6.0, 11.0, 16.0, 21.0),
…

We have reached the fixed point (1, 6, 11, 16, 21). Observe that the entries are yet again in A.P.

Conclusion

In each instance we find convergence to a tuple whose entries are in A.P. So if the seed is the *n*-tuple (a,x0,y0,…,b) then, writing d = (b - a)/(n - 1), the orbit appears in each case to converge to the tuple

$$(a,a + d,a + 2d,\ldots,a + (n - 2)d,b),$$

whose ith entry is a + (i - 1)d.

The reader is invited to investigate the matter further.

Project work

Instead of working with tuples, what if we work with MATRICES? The averaging now takes place in a higher dimension; given a matrix, we replace each element by the simple average of its immediate neighbours, with "neighbourliness" defined as follows: two elements are neighbours if they share the same row and lie in adjacent columns, or if they lie in the same column and lie in adjacent rows. Thus, for a matrix A, the elements ai,j and ar,s are neighbours if either i = r and j - s = ±1, or if j = s and i - r = ±1. (Stated more compactly: elements ai,j and ar,s are neighbours precisely when (i - r)2 + (j - s)2 = 1.) Observe that comer elements have two neighbours, border elements not lying at a comer have three neighbours, and "interior"elements have four neighbours.

Another possibility is: the comer elements could be left as they are, unchanged, and the averaging is done only for the remaining elements. There are clearly many such possibilities. Each would give rise to its own iteration.

Will we find the same behaviour as in the one-dimensional case? That is, will we find convergence to a matrix whose entries form an A.P in some sense? This question could form the basis for an interesting project. The reader is urged to investigate the matter further.

Remark. Various physical models can be connected with this iteration; e.g.,the numbers in the matrix could be the temperatures at various points of a conducting plate, and the averaging process could be a representation of the way in which the temperature equalizes itself through heat exchange.

Chapter 13

Geometric Iterations

13.1 Introduction

In this chapter the focus of study, which till now has been confined to arithmetic and algebra, shifts to geometry. We start by giving a few illustrations that show how iterations arise in geometry.

Reflections. Let f denote reflection in a line ℓ (so that ℓ acts like a mirror). Then $f \circ f = I$, because two successive reflections in the same line take us right back to the starting point; here I denotes the identity operation, which in geometry means "*do nothing, let the point stay where it is*". So reflection in a line is a 2-cycle function.

Half-turns. Let H denote a half-turn (a rotation through 180◦) about a fixed point O; then $H \circ H = I$ (for two successive half-turns about the same point are equivalent to a 360◦ turn and therefore to the identity map I). So a half-turn too is a 2-cycle function.

On the other hand, two successive half-turns about two distinct points, P and Q, are equivalent to a displacement through the vector $2\overrightarrow{PQ}$.

More generally, if θ is any angle ($\theta \neq 0^\circ, \theta \neq 360^\circ$) then the effect of two successive rotations f,g, with f centered at P and through an angle θ, and g centered at Q and through an angle $-\theta$, is a displacement. This fact comes of use when we have to move heavy furniture!

Quarter-turns. Let Q denote a quarter-turn about a fixed, point O; then $Q \circ Q = H$ and $Q(4) = H \circ H = I$. Here we have an example of a 4-cycle function.

An identity involving half-turns. For any quadrilateral ABCD, let P,Q,R,S be

the midpoints of the sides AB,BC,CD,DA respectivey. Then we have the curious relation

$$HP \circ HQ \circ HR \circ HS = I.$$

i.e., $HP \circ HQ \circ HR \circ HS(X) = X$ for all points X.

Frieze patterns. Here is an illustration from art. Let a design $\mathcal{B}$ (a "motif") be selected, and let the sequence $\langle \mathcal{B}; f \rangle$ be sketched for some appropriate function f. This simple action can result in rich geometrical patterns, depending on the choice of $\mathcal{B}$ and f. Many of the famous artifacts of Islamic art (e.g., those found in the Alhambra fortress in Granada, Spain, or in parts of the Islamic world) can be studied and appreciated from this perspective.

We give an example of such a sequence. Choose the motif to be the clubs symbol, ♣, and let f be, '*move the motif one unit to the right*'. Then the sequence $\langle ♣; f \rangle$ is an infinite strip:

$$\cdots ♣♣♣♣♣♣♣♣♣ \cdots$$

Designs of this type are known as frieze patterns, and they are often seen on temple walls, saree borders, pullovers, and so on.

Wall-paper patterns and crystals. The idea of a repeating one-dimensional pattern can easily be extended to two dimensions, resulting in wall-paper patterns. The three-dimension version of this is obviously a crystal; here a structure gets repeated in some definite manner in three dimensions. Other objects not typically referred to as crystals may possess such repetitive structures; for example, the DNA double helix, and in general other spiral forms that occur in nature.

We now study a few iterations involving geometric themes; *en route* we shall discover some pretty mathematics!

13.2 Perpendiculars

Let $\triangle ABC$ be acute-angled, and let P0 be a point on BC. We locate another point P1 on BC (i.e., on the same side) by dropping perpendiculars as follows: P0Q, from P0 to *AB*; QR, from *Q* to CA; and then RP1, from R back to BC. The position of P1 clearly depends on that of P0, so we write P1 = f(P0), where f is a function that 'acts' on the points on line BC, permuting them

amongst themselves (see Figure 13.1). We regard the movement from P0 to P1 as the first round of an iteration.

We now repeat this sequence of actions: we drop perpendiculars in turn to AB, to CA, to BC,…, and obtain the points

$$P2 = f(P1), P3 = f(P2), P4 = f(P3), \ldots.$$

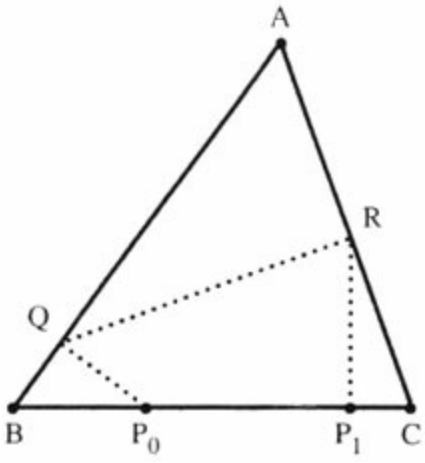


Figure 13.1. *After 1 round*

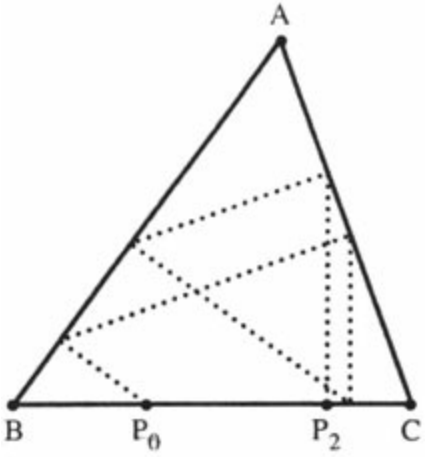


Figure 13.2. *After 2 rounds*

with the points P2,P3,P4,… all lying on BC. Figure 13.2 shows the path taken during the first two rounds of the iteration.

After a few.iterations, we notice something curious—*the path starts* to *close in on itself*. More precisely, the path converges to a triangular path inscribed in $\triangle ABC$. Note that this means that the sequence of points $\langle P0,P1,P2,P3,\ldots\rangle$ tends to some limiting position on BC. Figure 13.3 displays the limiting path

as an inscribed triangle (shown in a thick line). Observe that the inscribed triangle is similar to the original triangle (its sides are perpendicular to the sides of the original triangle, so the angles of the two triangles have the same measures; this explains the similarity).

It is natural to ask what happens when we change the position of P. Experimentation reveals that we reach the very same inscribed triangle each time! Figure 13.4 shows the path followed for another choice of P.

You may wonder what happens if the foot of the perpendicular lies not on the side on which it is dropped, but on its extension. The answer is that it does not matter. In Figure 13.5 the choice of figure is such that for the first few iterations, the feet of the perpendiculars lie on the extensions of the sides. After a few iterations, however, the situation "corrects" itself: the feet of the perpendiculars once again lie upon the sides, and we reach the very same inscribed triangle.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Till now our iteration has followed a clockwise direction: we dropped a perpendicular first to AB, then to CA, then to BC,…. But we could also opt for an anticlockwise direction, dropping a perpendicular first to CA, then to AB, then to BC,…. What happens if we do it this way?

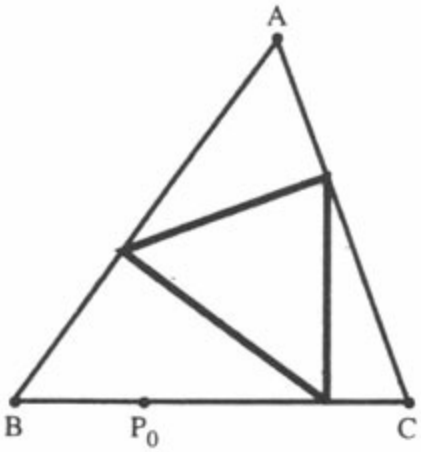


Figure 13.3. *Limiting path–1*

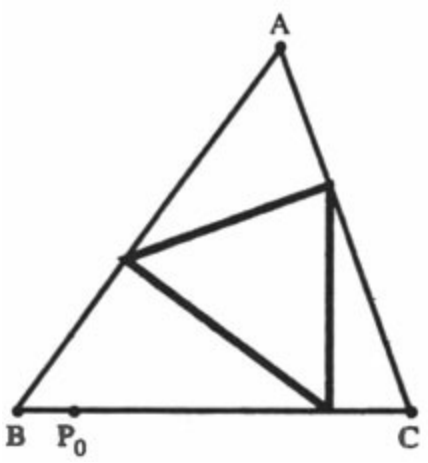


Figure 13.4. *Limiting path–2*

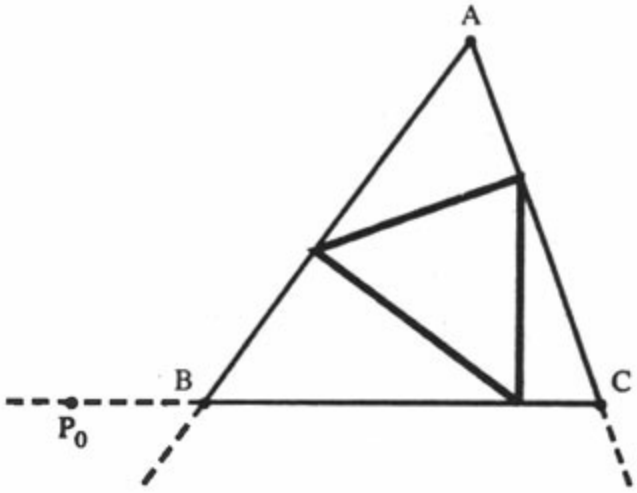


Figure 13.5. *Starting from a "wrong point"*

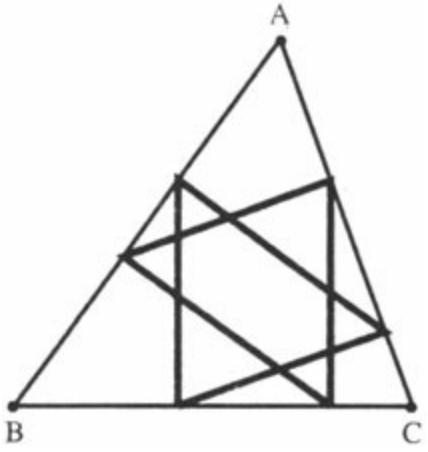


Figure 13.6.

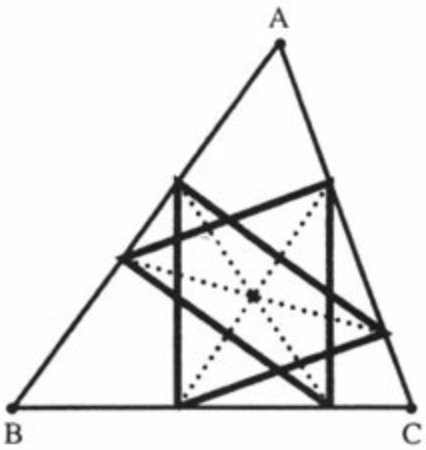


Figure 13.7.

Not surprisingly, we reach an inscribed triangle yet again; but, most pleasingly, the triangle turns out to be congruent to (and yet distinct from) the one got earlier! (See Figure 13.6.)

A surprising connection

An examination of the figure reveals that corresponding sides of these two triangles are parallel to one another (this owes to the perpendicularity relations noted earlier). Pairs of such objects are said to be *homothetic* to one another. If we draw the joins of corresponding pairs of points belonging to homothetic figures, the joins meet in a point called the *homothetic center* of the two figures. For example, an enlargement mapping yields pairs of homothetic figures, and the center of homothety is simply the center of the enlargement. A half-turn also yields pairs of homothetic figures, and the center of homothety in this case is the center or "pivot" of the half-turn ; it bisects the joins of corresponding pairs of points. (Comment: A half-turn is the same as an enlargement with scale factor equal to –1. Note that a half-turn is the only homothety in which the input figure is congruent to the output figure.)

In the current example the two figures are congruent to one another, so the homothety is a half-turn . The homothetic center turn s out to be a "known" point— it is the *symmedian point* of the triangle. The point is constructively defined as follows: we locate G, the centroid of the triangle, draw the medians GA,GB and. GC, then reflect them, respectively, in the lines internally

bisecting the angles at A,B and C. The reflected segments are known as *symmedians*, and they meet in the symmedian point of the triangle.

Another way of finding the same point is as follows: draw the circumcircle of triangle ABC, then draw tangents to the circle at A,B and C. Let the triangle enclosed by the tangents be XY Z, with the vertices labelled so that X is opposite A (i.e., the tangents at B and C meet at X), Y is opposite B and Z is opposite C; then the lines AX, *B*Y and CZ meet in the symmedian point.

Exercises

13.2.1 Prove that the inscribed triangle referred to above is similar to the original triangle.

13.2.2 What happens if the original triangle is right-angled? How many iterations are needed in this case to reach the limiting path?

13.2.3 What happens if the original triangle is obtuse-angled?

13.2.4 What happens if we work with a convex quadrilateral instead of a triangle?

13.2.5 Prove that the sequence

$$\langle P, f(P), f(f(P)), \ldots, \rangle$$

converges to a limiting point, which moreover is the same for all points P on BC. (Hint: Use trigonometry to find the distances from B to the sequence of iterates of P.)

Explanations

We now justify the claim made that the two inscribed triangles obtained "in the limit" above are congruent and homothetic to one another, the center of homothety being the symmedian point. Our proof will be based on computation.

We note firstly that the sides of the inscribed triangles are respectively perpendicular, in some order, to the sides of the parent triangle, implying that each triangle is similar to the parent triangle. Therefore, the two triangles are similar to one another; and homothetic too, as their sides are parallel to one

another.

In Figure 13.8, the two inscribed triangles are labelled A′B′C′ and A″B″C″, respectively; for convenience, we refer to them as T′ and T″, respectively, and to △ABC as T. The labelling of T′ and T″ has been done in the obvious manner, to draw attention to their similarity (note how their angles correspond to one another). Thus we have angle equalities such as

$$\angle B'A'C' = \angle BAC = \angle B''A''C'', \angle A'B'C' = \angle ABC = \angle A''B''C''$$

and so on.

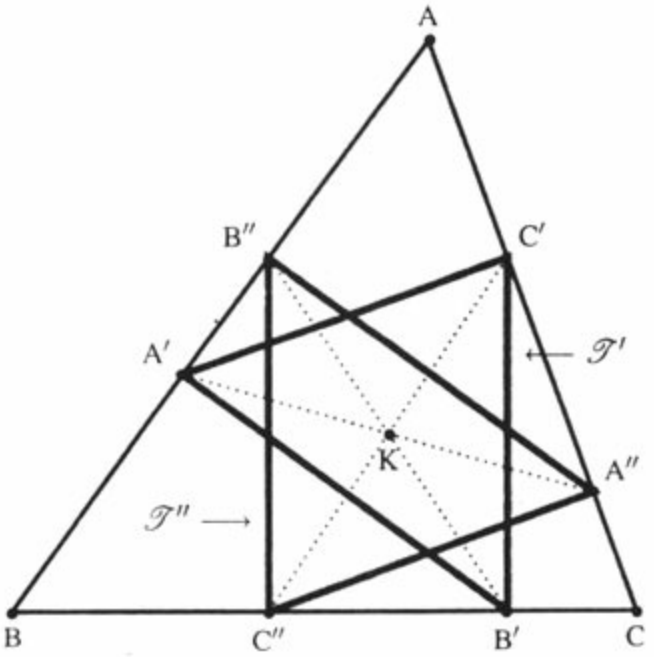


Figure 13.8. *Study of the two homothetic triangles*

We first show that triangles T′ and T″ are congruent to one another. Write a′,b′,c′ for the sides of T′, and a″,b″,c" for the sides of T″. Using elementary trigonometry we obtain the following relations:

$$a = BC = BB' + B'C = c' \sin B + a' \tan C'$$

$$b = AC = AC' + C'C = b' \tan A + a' \sin C,$$

$$c = AB = AA' + A'B = b' \sin A + c' \tan B.$$

The relations may be displayed in matrix form thus:

$$\cot C \; 0 \; \csc B \quad \csc C \; \cot A \quad 0 \; 0 \; \csc A \quad \cot B \cdot a' \; b' \; c' = a \; b \; c \; .$$

Using well known formulas such as

$$\cos A = b2 + c2 - a2\ 2bc\ ,\sin A = a\ 2R,$$

where R is the radius of the circumcircle of the triangle, and doing a bit of simplification, the equations take the form shown below (here $\Delta = abc/4R$ denotes the area of the triangle):

$$1\ 4\Delta \cdot a2 + b2 - c2\ 0\ 2ca\ 2ab\ b2 + c2 - a2\ 0\ 0\ 2bc\ c2 + a2 - b2\ \cdot a'\ b'\ c' = ab\ c$$

The equations may be solved for a', b', c', and we get a most pleasing result:

$$a'\ b'\ c' = 4\Delta\ a2 + b2 + c2 \cdot ab\ c$$

Using the (certainly not too well known!) fact that

$$4\Delta\ a2 + b2 + c2 = \tan\omega,$$

where ω is the Brocard angle of the triangle, we may write this as

$$a'\ b'\ c' = \tan\omega \cdot ab\ c$$

The symmetry of the result tells us immediately that the very same equations will be obtained for a'', b'', c''. It follows that $a' = a'', b' = b''$ and $c' = c''$, and the congruence of the two inscribed triangles follows.

As segments $B''C''$ and $C'B'$ are congruent and parallel to one another, their center of homothety, K, lies at the center of rectangle $B''C''B'C'$. This means in particular that the perpendicular distance of K from side BC is

$$1\ 2a' = 1\ 2\tan\omega\ \cdot\ a = ka, \quad \text{where } k = 1\ 2\tan\omega.$$

We find similarly that the distances of K from sides CA and AB are kb and kc respectively. The ratios of the distances of K from the sides BC,CA,AB are therefore as a : b : c. But this is precisely the property which characterizes the symmedian point! The claim we made earlier therefore stands justified.

13.3 Angle bisectors

The iteration we study next deals with angle bisectors. We start by defining the relevant terms.

Incenter of a Triangle The internal angle bisectors of a triangle meet in a point—the *incenter* of the triangle, or the center of the circle that fits snug

within the triangle. The circle is the *incircle* of the triangle. (To see why the bisectors meet in a point, use the fact that each point on an angle bisector is equidistant from the two arms of the angle; see Figure 13.9.)

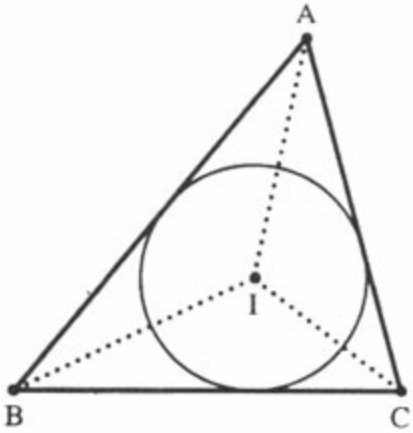


Figure 13.9.

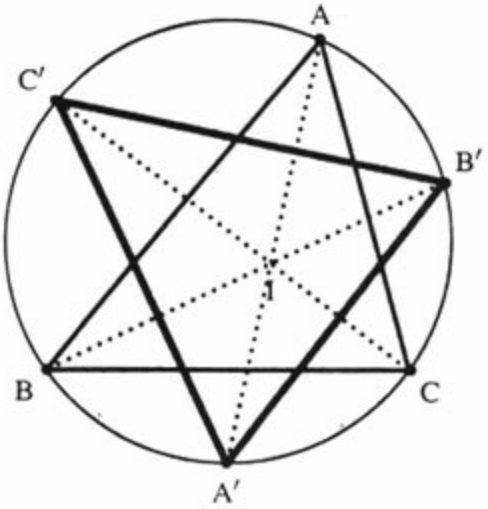


Figure 13.10.

The iteration

Let A,B,C be three distinct points on a circle Γ, and let $\triangle ABC$ be called T0; this triangle will serve as the seed of the iteration. Let the angle bisectors of T0 be drawn. and let them be extended beyond the sides of the triangle to meet the circle Γ. Let T1 be the triangle whose vertices are the three points of intersection so formed. (See Figure 13.10; the angle bisectors are shown in a dotted line; T1 is $\triangle A'B'C'$.)

Take the actions done so far to be one round of an iteration f; that is, take T1 = f(T0) to be the output produced with T0 as input. Let T2,T3,… be computed as follows:

$$T2 = f(T1), T3 = f(T2), T4 = f(T3), \ldots.$$

(Thus, to get T3 from T2, the angle bisectors of T2 are drawn and extended till they meet Γ. The three points of intersection are the vertices of T3; and so on.) What happens now? Figures 13.11 and 13.12 show the results of the iteration after 5 and 10 rounds respectively.

Observations

An interesting fact emerges when we examine these sketches: as we continue the iterations, *the triangles start to look more and more equilateral in shape*, and this happens regardless of the shape of the original triangle. Figures 13.13 shows the result of 10 rounds of the iteration using a differently shaped triangle as seed; the original triangle is shown in a light line and the final triangle in a thick line.

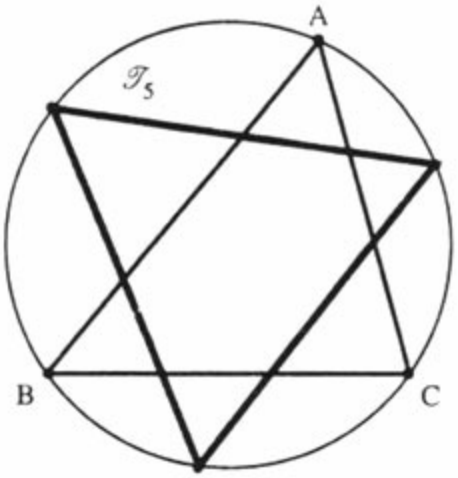


Figure 13.11. *After* 5 *rounds*

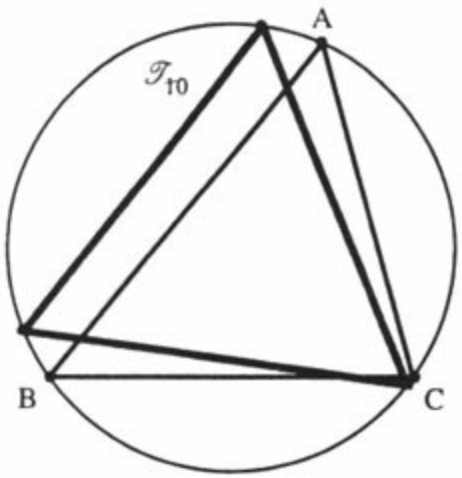


Figure 13.12. *After* 10 *rounds*

Note that the triangle itself does not stay fixed: T100 is distinct from T101, but they are both close to being equilateral. But T100 and T102 are "almost" the same triangle (they very nearly coincide), and likewise for T101 and T103.

Analysis

To see why convergence takes place, we compute the angles of T1 and check whether they are closer to one another than the angles of T0. This is easy to do.

In Figure 13.14, T0 is $\triangle ABC$ and T1 is $\triangle DEF$. We compute the angles of T1 in terms of the angles of T0, using the well known theorem concerning "angles in the same segment." (If P,Q are points on an arc XY of a circle, then $\angle XPY = \angle XQY$.) We argue as follows. We have, $\angle FDE = \angle FDA + \angle ADE$. Next,

$$\angle FDA = \angle FCA = C\ 2\ , \angle ADE = \angle ABE = B\ 2\ ,$$

therefore $\angle FDE = 1\ 2(B + C) = 90\ -1\ 2A$. Similarly we get, $\angle FED = 90\ -1\ 2B$ and $\angle DFE = 90\ -1\ 2C$. Therefore the angles of $\triangle DEF$ are

$$90^\circ\text{-}A\ 2\ , 90^\circ\text{-}B\ 2\ , 90^\circ\text{-}C\ 2\ .$$

(Check that the angle sum is 180°.) So we have an iteration on triples that acts as follows:

$$(x,y,z) \mapsto 90\ -x\ 2, 90\ -y\ 2, 90\ -y\ 2.$$

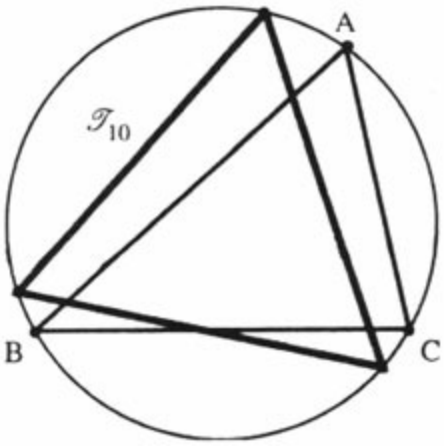


Figure 13.13. *A different* T0

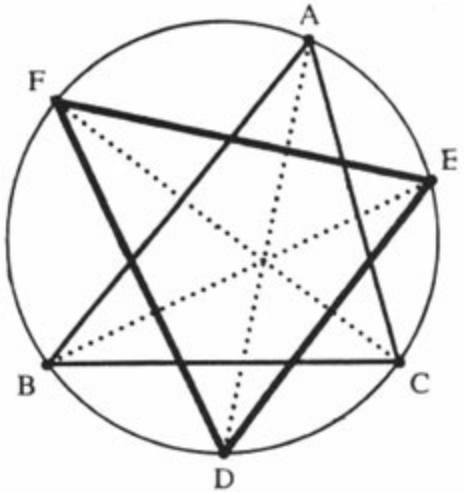


Figure 13.14. *Angle computations*

Consider the function f given by f(x) = 90 -1 2x. Let any number x0 be chosen, and let its f-orbit ⟨x0;f⟩ be computed. Does the sequence converge to a limit? We do a few trials to find out.

- x0 = 20 60.31,59.84,… The sequence produced is: 20, 80, 50, 65, 57.5, 61.25, 59.38,
- x0 = 40 60.16,59.92,… The sequence produced is: 40, 70, 55, 62.5, 58.75, 60.63, 59.69,
- x0 = 5 59.14,60.43,59.78,… The sequence produced is: 5, 87.5, 46.25, 66.88, 56.56, 61.72,

The figures clearly suggest that regardless of the seed chosen, the sequence

converges to a limiting value of 60. Indeed this is so, and the exercise set below asks you to find a proof. As the iteration is the same for each angle of the triangle, it follows that each angle converges to 60◦, and so the triangles do indeed converge to an equilateral shape.

Degree of 'non-equilateralness' of a triangle

Here is one way (the exercise set asks you to find another way!) of showing that the f-orbit of any starting number converges to a limiting value of 60. We have already seen that if T0 has angles x◦,y◦,z◦, then T1 has angles

$$90 \text{ -x } 2^\circ,\ 90 \text{ -y } 2^\circ,\ 90 \text{ -z } 2^\circ$$

Consider the *differences* between the angles of T0; they are

$$|x - y|^\circ, |y - z|^\circ \text{ and } |z - x|^\circ$$

respectively. In the case of T1, the differences come out to be

$$|x - y|^\circ\ 2\ ,\ |y - z|^\circ\ 2 \text{ and } |z - x|^\circ\ 2\ ,$$

respectively. *So the differences have all been halved.*

Given any triangle, the extent to which it is *not* equilateral can be measured; it seems reasonable, surely, to describe a triangle with angles of 55◦,60◦,65◦ as "closer to being equilateral" than one with angles of 10◦,60◦,110◦. Given a triangle T with angles α,β,γ, a possible index for measuring how close it is to being equilateral is the quantity δ (T) given by

$$\delta(T) = |\alpha - \beta| + |\beta - \gamma| + |\gamma - \alpha|.$$

Observe that $\delta \geq 0$ for all triangles, and moreover that $\delta(T) = 0$ if and only if $\alpha = \beta = \gamma$; that is, if and only if T is equilateral.

In terms of this measure, the result obtained above can be stated succinctly as:

$$\delta(T1) = 1\ 2\delta(T0).$$

Similarly, $\delta(T2,) = \delta(T1)/2$, and so on. Since the δ-value is halved each time, it follows that δ(Tn) tends to 0 as n increases indefinitely. so the T's do indeed tend towards the equilateral shape.

Exercises

13.3.1 Let I0,I1,I2,… be, respectively, the incenters of the triangles T0,T1,T2, …. Does the sequence

$$\langle I0,I1,I2,\ldots\rangle$$

possess a limit?—i.e., do the incenters converge to a point? If so, what is this point?

13.3.2 Find another proof for the claim that for the iteration function given by f(x) = 90 -1 2x, the f-orbit corresponding to any seed converges to 60.

13.4 Medians

From angle bisectors we move on medians. Here are two definitions.

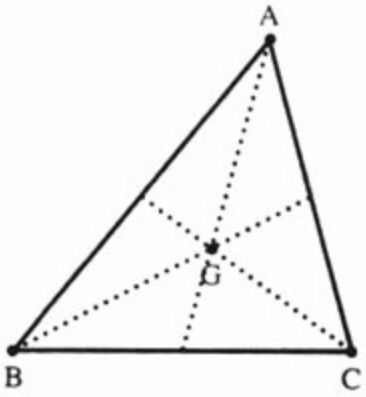


Figure 13.15.

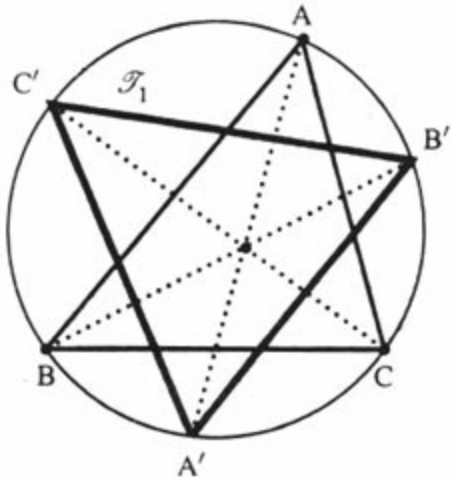


Figure 13.16.

Median of a Triangle; Centroid The line segment connecting a vertex of a triangle to the mid-point of the opposite side is called a *median* of the triangle. The three medians of any triangle meet in a point, called the *centroid* of the triangle; see Figure 13.15.

The iteration

We start with triangle T0, with vertices A,B and C, inscribed in a circle Γ. Let its medians be drawn, and let them be extended, as was done with the angle bisectors, to meet Γ. Let T1 be the triangle whose vertices are the three points of intersection; see Figure 13.16.

Let the actions performed so far be considered as one round of an iteration f; take T1 = f(T0) to be the output produced from the input T0.

Since T0 and T1 share the same circumcircle, the iteration can be continued. So we get

$$T2 = f(T1), T3 = f(T2), T4 = f(T3), \ldots,$$

and so on. (To produce T3 from T2, the medians of T2 are drawn and extended till they meet Γ; the three points of intersection are the vertices of T3.)

What happens now? Figures 13.17 and 13.18 show the results of the iteration after 5 and 15 rounds respectively. (In each case T0 has been drawn in a light line and the final Ti in a heavy line.)

Remarkably, we find exactly the same thing happening as in the angle-bisector iteration: *the triangles start to look more and more equilateral in shape*. And as earlier, this happens regardless of the shape of the seed triangle. Figure 13.19 shows the result after 15 rounds of the iteration, with a differently shaped triangle as seed.

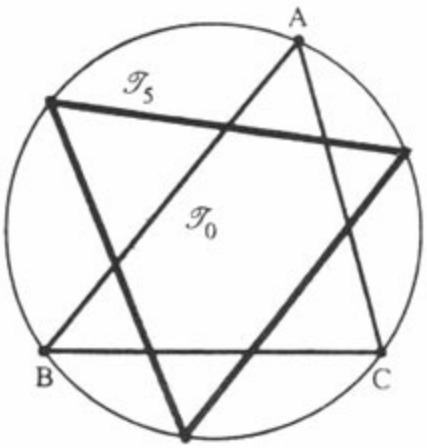


Figure 13.17. *After* 5 *rounds*

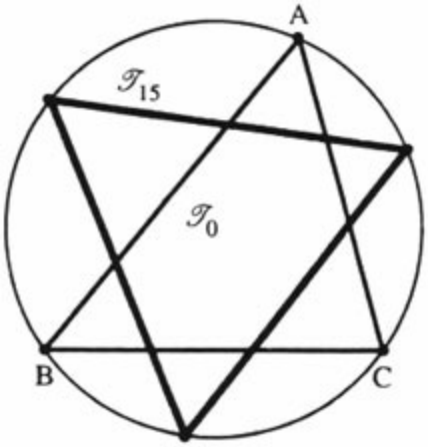


Figure 13.18. *After* 15 *rounds*

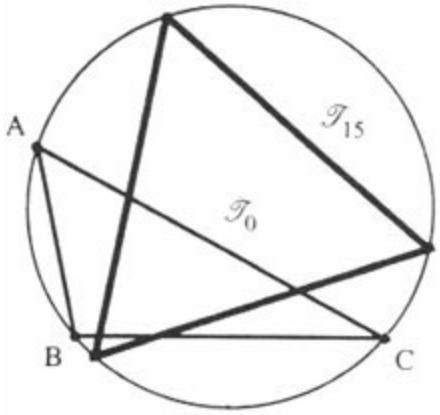


Figure 13.19. *After* 15 *rounds, with another* T0

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

In order to do an analysis of this problem, we must compute the angles of

T1 and check whether they are in some sense closer to one another than the angles of the parent triangle; or we could work with the sides rather than the angles. This was easy to do in the case of the angle-bisector iteration. Unfortunately the analysis in this case is rather more involved. The ambitious reader may want to try finding a proof independently, before proceeding to the analysis given at the end of this chapter (in the Appendix).

Exercises

13.4.1 Let Gi denote the centroid of Ti,i = 0,1,2,…, and consider the sequence

$$\langle G0,G1,G2,G3,\ldots\rangle.$$

Does the sequence possess a limit? That is, do the centroids converge to some limiting position?

13.4.2 Instead of drawing medians, we could draw lines according to the following scheme.

Let k denote a fixed number, $0 < k < 1$. Labelling T0 as ABC, we locate points P on BC,Q on CA and R on AB according to the rule

$$BP\ PC = CQ\ QA = AR\ RB = k,$$

draw the lines AP,BQ,CR, then extend them to meet the circle Γ; the three points of intersection are then the vertices of T1. Likewise we generate T2 from T1,T3 from T2, and so on.

Let Gi be the centroid of Ti. Does the sequence $\langle G0,G1,G2,\ldots\rangle$ converge to a point? Does convergence depend on the choice of k? Use a computer to do experiments and make a guess based on your findings. (Observe that k = 0.5 corresponds to the original iteration.)

Numerical analysis

As mentioned earlier, the analysis of the medians iteration is rather involved. We shall give the full details later. For now, we shall only show how the iteration can be described in algebraic terms; this then allows a numerical investigation.

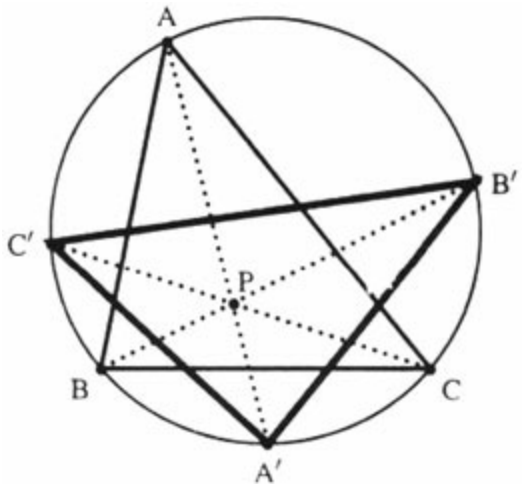


Figure 13.20. *A subsidiary result*: $B'C' : C'A' : A'B' = AP \cdot BC : BP \cdot CA : CP \cdot AB$

We first establish a subsidiary result. In Figure 13.20, we see a triangle ABC, inscribed in its circumcircle Γ, with an interior point P. The lines AP,BP and CP, when extended, meet the circumcircle again in the points A',B' and C', respectively. We shall show that

$$B'C' : C'A' : A'B' = AP \cdot BC : BP \cdot CA : CP \cdot AB.$$

The proof is as follows. From the chord theorem, we get:

$$AP \cdot PA' = BP \cdot PB' = CP \cdot PC'.$$

Let k be the common value of these three products. From the similarity of triangles $B'PC'$ and CPB, we get:

$$B'C'\ BC = PB'\ PC\ ,\ \therefore B'C = BC \cdot PB'\ PC = k \cdot BC\ BP \cdot CP = k \cdot AP \cdot BC\ AP \cdot BP \cdot CP.$$

It follows that $B'C' : C'A' : A'B' = AP \cdot BC : BP \cdot CA : CP \cdot AB$, as claimed.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

If P happens to be the centroid of triangle ABC, then the lengths AP,BP,CP are two-thirds of the lengths of the medians through A,B,C, respectively. Writing ma,mb,mc for the lengths of these three medians, and a',b,c' for the lengths of the sides of triangle $A'B'C'$, we see that

$$a' : b' : c' = am\ a : bmb : cmc,$$

certainly a pretty and compact result!

From **APOLLONIUS'S THEOREM** we know that

$$ma2 = b2\,2 + c2\,2 - a2\,4\ ,mb2 = c2\,2 + a2\,2 - b2\,4\ ,mc2 = a2\,2 + b2\,2 - c2\,4\ .$$

Using these relations, we are now able to make a numerical study of the convergence profile of the medians-iteration. Writing the relations $a' : b' : c' = ama : bmb : cmc$ as

$$a'2 : b'2 : c'2 = a2m\ a2 : b2m\ b2 : c2m\ c2,$$

we see that $a'2 : b'2 : c'2$ is the same as

$$a2(2b2 + 2c2 - a2) : b2(2c2 + 2a2 - b2) : c2(2a2 + 2b2 - c2).$$

It will be evident now that it is more convenient to deal with the *squares* of the sides of the triangle than with the sides themselves. Further, as it is only the *ratios* that the sides bear to one another that are of relevance to us, we define quantities x,y,z,x',y',z' as follows:

$$x = a2\,a2 + b2 + c2,y = b2\,a2 + b2 + c2,z = b2\,a2 + b2 + c2,$$

and

$$x' = a'2\,a'2 + b'2 + c'2,y' = b'2\,a'2 + b'2 + c'2,z' = b'2\,a'2 + b'2 + c'2;$$

then $x + y + z = 1 = x' + y' + z', x : y : z = a2 : b2 : c2$, and $x' : y' : z' = a'2 : b'2 : c'2$.

Now we describe what iteration "does" to the triple (x,y,z). From the above we see that the iteration maps (x,y,z) to the triple (x',y',z'), where

$$x' = x(2y + 2z - x)\ 4(xy + yz + zx) - (x2 + y2 + z2),$$

$$y' = y(2z + 2x - y)\ 4(xy + yz + zx) - (x2 + y2 + z2),$$

$$z' = z(2x + 2y - z)\ 4(xy + yz + zx) - (x2 + y2 + z2).$$

The common denominator here is the sum

$$x(2y+2z-x)+y(2z+2x-y)+z(2x+2y-z) = 4(xy+yz+zx)-(x2+y2+z2).$$

The important consequence of this formulation is that we have at hand a fully numerical version of the iteration. We now test it out.

Example 1. Consider a triangle with sides a = 3,b = 4,c = 5. As a2 : b2 : c2 = 9 : 16 : 25 and 9 + 16 + 25 = 50, the starting triple is (x0,y0,z0) = (0.18,0.32,0.50). The subsequent triples, computed via the formula displayed above, are:

(0.1800, 0.3200, 0.5000), (0.3108, 0.3936, 0.2956),

(0.3374, 0.3279, 0.3347), (0.3333, 0.3333, 0.3333),

(0.3333, 0.3333, 0.3334), (0.3333, 0.3333, 0.3333).

The speed with the triple (0.3333, 0.3333, 0.3333) (which corresponds to the equilateral triangle) is reached is most impressive! (There is a small amount of round-off error in the displays: the last two triples sum to 0.9999 rather than to 1.)

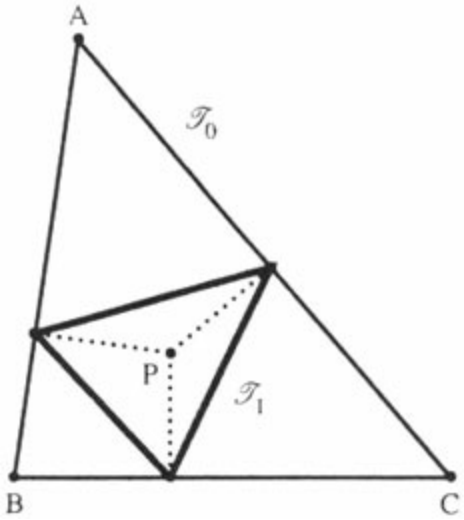


Figure 13.21. *The pedal triangle*

Example 2. Now consider a triangle with sides a = 2,b = 5,c = 6. We get the following result:

(0.0615, 0.3846, 0.5538), (0.1789, 0.5210, 0.3001),

(0.3194, 0.2778, 0.4028), (0.3410, 0.3322, 0.3268),

(0.3333, 0.3334, 0.3333), (0.3333, 0.3333, 0.3333).

Once again the swift rate of convergence is in full view.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

The full proof that the sequence of iterates converges to an equilateral shape is given at the end of the chapter.

13.5 Pedal triangles

We start with an acute angled triangle T0, with vertices A,B,C, and a point P lying within the region enclosed by the triangle. Let perpendiculars be dropped from P to each side of T0, and let the feet of these perpendiculars be the vertices of a new triangle T1. This triangle is the *pedal triangle* of P relative to T0. Note that T1 is "inscribed" in T0, for its vertices lie on the sides of T0, one to each side (see Figure 13.21).

As P lies within T1, we can repeat these actions and construct the pedal triangle of P relative to T1, by dropping perpendiculars from P to the sides of T1 and joining the feet of the perpendiculars to one another. In this manner we obtain another triangle T2, and P lies within this triangle too. We now construct the pedal triangle of P relative to T2, obtain another triangle, T3, and we continue thus, obtaining a sequence ⟨T0,T1,T2,T3,…⟩, in which each Ti is inscribed within the preceding one. The triangles shrink in size very rapidly, and soon the eye is unable to distinguish the triangles from the point P itself. Figure 13.22 shows the T's till T3.

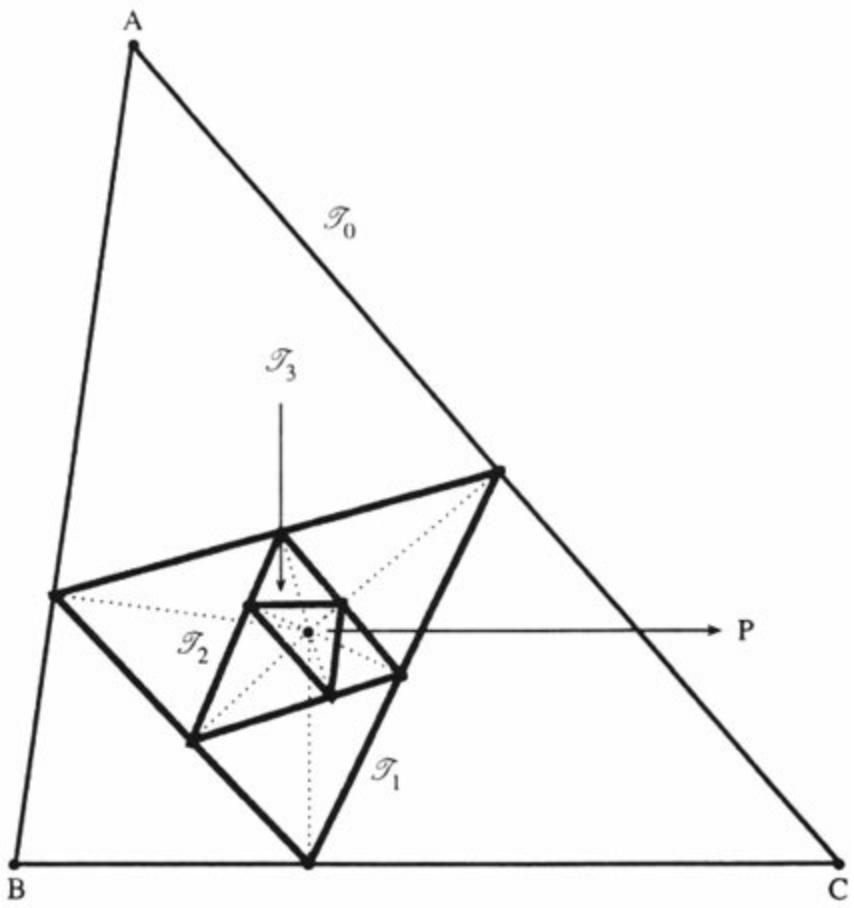


Figure 13.22. *Inscribed pedal triangles*

There is a nice feature about the sequence that could easily escape the eye: T3 is similar to T0! Examine the figure carefully; you should be able to see the similarity on your own. (The two triangles are in fact homothetic to one another, and P is the center of homothety.)

Exercises

13.5.1 Prove that T3 is similar to T0. (*Hint*. Compute the angles of T1. Do not draw the pedal triangles after T1, else the figure becomes too confusing!)

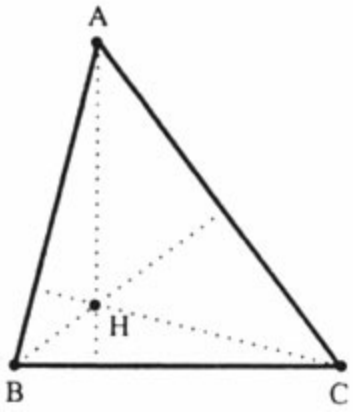


Figure 13.23.

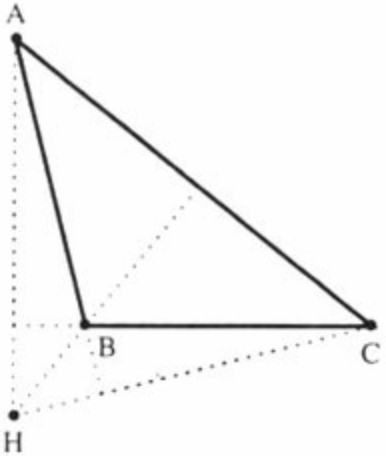


Figure 13.24.

13.6 Altitudes

Having studied iterations associated with the angle bisectors and the medians of a triangle, it is natural to consider the case of altitudes. The iteration turn s out to be quite exotic, producing complete chaos! The earlier iterations were both fairly "well behaved." in terms of yielding predictable results, but that is not the case here. Before we start we need two definitions.

Altitude, Orthocenter The line drawn from a vertex of a triangle perpendicular to the opposite side is called an *altitude* of the triangle. The three altitudes of a triangle always meet in a point, called the *orthocenter* of the triangle. The orthocenter can lie either inside or outside the triangle (see Figures 13.23 and 13.24).

The iteration

We start the iteration with triangle ABC inscribed in a circle Γ; call the triangle T0, and let its orthocenter be H0. The altitudes of T0 are drawn and extended till they meet Γ. Let T1 be the triangle whose vertices are the three new points of intersection, and let H1 be its orthocenter. Take T1 to be the output arising from the input T0.

We now iterate these actions: we draw the altitudes of T1, extend them till they meet Γ, and join the three points of intersection to one another; then the resulting triangle is T2, and its orthocenter is H2. We continue in this manner, obtaining a sequence of triangles,

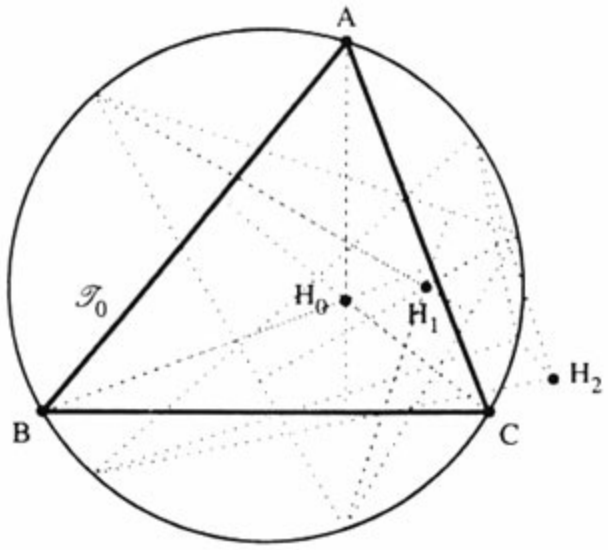


Figure 13.25. *Altitude iteration, after* 2 *rounds*

$$\langle \text{T0,T1,T2,T3,}\ldots\rangle,$$

and a corresponding sequence of orthocenters,

$$\langle \text{H0,H1,H2,H3,}\ldots\rangle.$$

Having studied in some detail the iterations with angle bisectors and medians, we may now be tempted to guess that the sequence of orthocenters will converge to a point. But we are in for a surprise!

Figure 13.25 shows the result after 2 rounds of the iteration (the orthocenters are shown, but some details have been hidden so as to avoid a visual mess).

Figure 13.26 shows the result after 25 rounds of the iteration (the labelling of the Hi's has been suppressed, for obvious reasons; likewise for the construction lines used). Note the dust-like scatter.

Figure 13.27 shows the result after 85 rounds. There appears now to be a random scatter! (There is a curious star-like effect, with streaks in six directions symmetrically placed about the circle; but this is hard to explain.) If we continue the iterations still further, we find a circular region entirely covered with dots. A careful analysis reveals that the radius of this circle is three times the radius of the circumcircle of T0.

Remark. For some choices of the starting triangle T0, a dust-like effect is *not* obtained; instead, we get a cycle. The reader may want to investigate under what conditions this will happen.

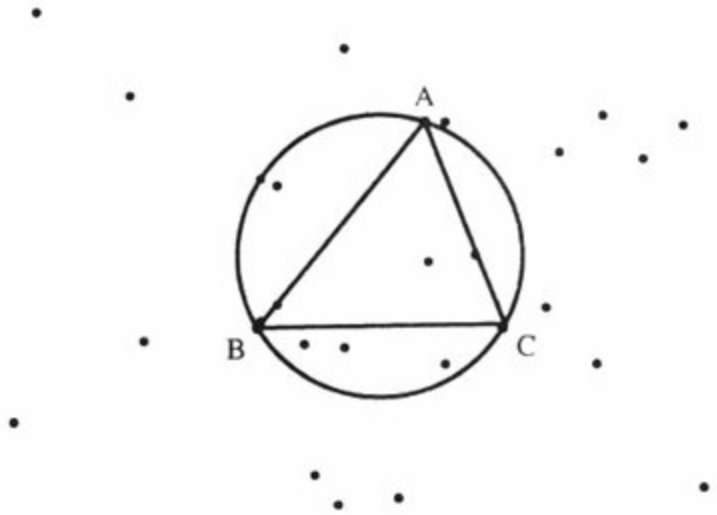


Figure 13.26. *After* 25 *rounds*

Exercises

13.6.1 For which triangles does the orthocenter lie (i) within the triangle (ii) on one side of the triangle (iii) one side of the triangle? Can the orthocenter ever lie in the interior of a side of the triangle?

13.6.2 Show that the altitudes of a triangle meet in a point.

13.6.3 In the iteration just considered, do the Ti's converge to any limiting shape?

13.6.4 Consider all possible triangles inscribed in a circle Γ, and let $\mathcal{H}$ denote the set of their orthocenters. Show that $\mathcal{H}$ occupies a circular region whose radius is three times that of Γ. You will need to use the following well known result due to Euler:

> *For any triangle, the circumcenter O centroid G and orthocenter H lie on a straight line the "Euler line" of the triangle, with H between O and G and moreover,* OH = 3OG.

13.6.5 It had been noted above that "for some choices of the starting triangle T0, a dust-like effect is *not* obtained; instead, we get a cycle." Investigate under what conditions this will happen.

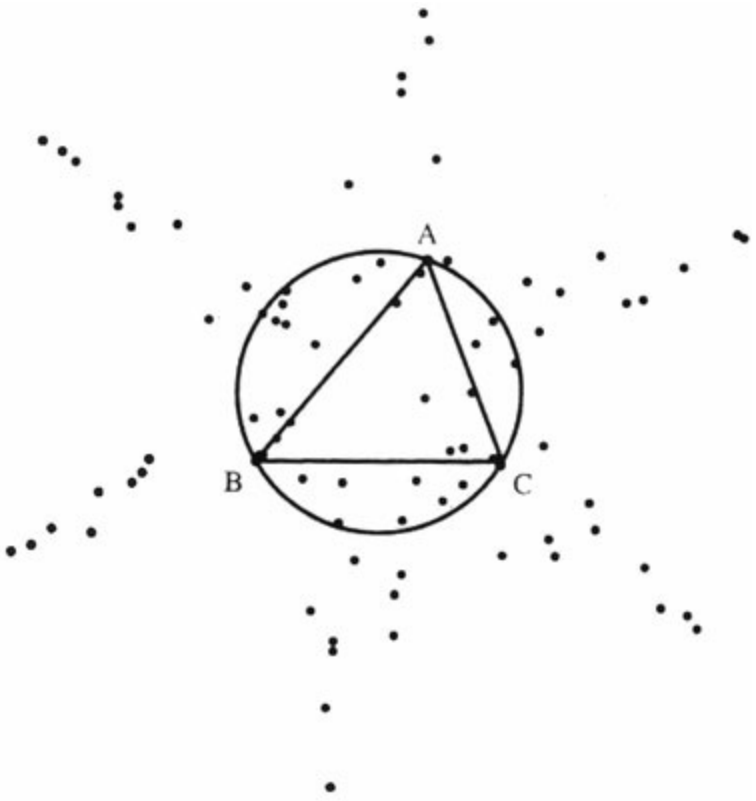


Figure 13.27. *After* 85 *rounds*

13.7 Pretty pictures

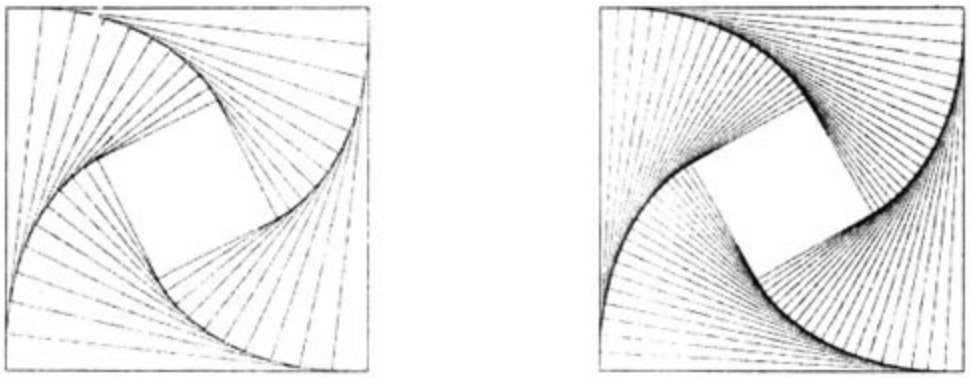
Next we show how simple iterative actions can give rise to very attractive pictures.

13.7.1 Nested Squares

Let k be a number lying between 0 and 1. Let S0 be a square, and let its vertices be labelled in cyclic order as A,B,C,D. Let points P,Q,R,S, respectively, be located on the sides *AB*, BC,CD,DA of S0, according to the following rule:

$$AP\ AB = BQ\ BC = CR\ CD = DS\ DA = k.$$

(If k = 0.5, then P,Q,R,S are the mid-points of the sides of S0.) The segments PQ,QR,RS,SP are drawn, giving another square, S1, which we take to be the output of a function with input S0. Observe that S1 is smaller than S0 and is wholly contained within it; it has the same center as its parent square, but has been turn ed through a small angle. To draw S2 we temporarily rename S1 as *ABCD* and locate points P,Q,R,S on its sides as per our rule; then S2 is the square *PQRS*;



k = 0.1, 10 *iterations* *k* = 0.05, 20 *iterations*

Figure 13.28. *Nested squares*

and so on. We thus obtain a sequence of squares

$$\langle S0,S1,S2,S3,\ldots\rangle,$$

in which each square lies inside all the preceding ones. It is easy to see that the squares converge to the center of symmetry of the square. The picture that results is a pretty one; two such pictures have been portrayed in Figure 13.28.

Exercises

13.7.1 Prove that Si is a square for each i.

13.7.2 What picture results when k = 0.5?

13.7.3 Which values of k generate the nicest pictures? (You will need to do some experimentation to find out!)

13.7.4 If you work with non-square quadrilaterals instead of squares, what do you find?

13.7.5 If you carefully study the pictures generated above, you will see many spiralling curves. This is actually an optical effect: even though we have not drawn any curve, it so happens that the lines that we have drawn are all tangent to a particular curve, and it is this curve that we "see. So in this sense the curve is not merely an illusion but has been created by the lines. (The lines are said to *envelope* the curve.)

List some daily life situations in which straight lines create the effect of a curve. (Hint. Think of the "tea-cup curve.")

13.7.6 Four men (A,B,C,D) are stationed at the comers of a square, one at each vertex, and at a given instant they start walking towards one other: A towards B,B towards C,C towards D and D towards A, all at the same speed. As they are all in movement, each one must continuously alter his direction of movement. What paths do the men trace out? Where do they meet?

13.7.7 A dog spots a rabbit and starts to chase it; the rabbit heads straight for its burrow. What sort of path does the dog follow? (Experiment with different possibilities. If the dog and rabbit run in the same direction, then the path is simply a straight line. What happens if the rabbit initially starts in a direction which is perpendicular to the dog's path?)

13.7.2 Nested Triangles

We do exactly the same thing with triangles. Let T0 be a equilateral triangle and let k be any number lying between 0 and 1. We label the vertices of T0 as A,B,C, then locate points P,Q,R on the sides of T0 using the following rule:

$$BP\ BC = CQ\ CA = AR\ AB = k;$$

then we declare T1 to be $\triangle$PQR. We find that T1 is equilateral, but of smaller size that T0; it has the same center of symmetry as its parent triangle, and is wholly contained within it. We similarly generate T2 from T1,T3 from T2, and so on. The sequence

$$\langle T0,T1,T2,T3,\ldots\rangle$$

then consists of nested triangles, one within the other, converging to the center of symmetry of the triangle. Figure 13.29 shows two sample pictures.

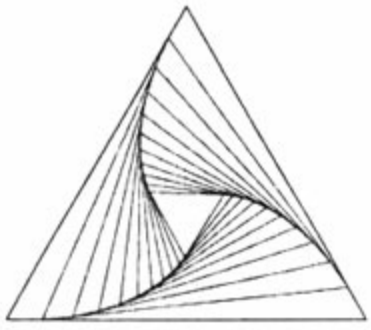
This example and the previous one ("nested squares") form the basis of,many pretty *Nail-and-thread* art pieces that can be found in the craft center of a school.

Exercises

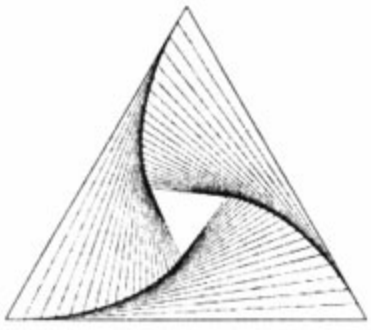
13.7.8 Suppose that T0 is not equilateral. What can be said about T1,T2,T3, … ? Are the triangles further down the sequence 'more equilateral' than the earlier ones?

13.8 Fractals

Fractals constitute a relatively modem discovery; they date to the latter half of the 20th century and may be said to have originated in the work of the French mathematician and geometer Benoit Mandelbrot. In the bargain a new class of mathematical objects has emerged, one that holds considerable relevance to the study of the natural world. The concept is a relatively simple one, but it is remarkable how far it takes us. We illustrate the idea with a simple example —that of the *snowflake curve*.



$k = 0.1$, 10 *iterations*



$k = 0.05$, 20 *iterations*

Figure 13.29. *Nested triangles*

Figure 13.30. *Action of* f *on a line segment*

★★★★★★★★★★★★★★★★★★★★

We first describe how the function f acts. Let a line segment $\ell = AB$ be given, and let f act on ℓ thus: we locate the points X,Y which trisect AB (with X nearer to A than to B, so the points occur on ℓ in the order A,X,Y,B), and then locate Z such that $\triangle XYZ$ is equilateral. The ambiguity regarding the side of ℓ on which Z is located is resolved thus: in $\triangle XYZ$, the vertex order (X,Y,Z) is required to be *anticlockwise*. Following this, segment XY is deleted and replaced by segments XZ and Y Z (see Figure 13.30).

Now let a geometric figure G0 be given, composed only of line segments; for instance, G0 could be a polygon. We let f act on each edge of G0, and obtain a new geometric figure, say G1. The same treatment is given to G1, and so on, and we thus obtain a sequence of geometric figures,

$$\langle G0,G1,G2,G3,\ldots\rangle,$$

each of which is composed only of line segments, which get smaller and smaller as we go down the sequence. The following sequence of figures shows what happens when G0 is an equilateral triangle. Note the beauty of the figures! It is possible in the minds' eye to imagine what happens 'in the limit', but it is impossible to draw it. The limiting object is known as the *snowflake curve*— for obvious reasons, as this is what the object resembles it its broad outline and symmetry. (We have stopped at G5 because, as you will readily see, drawing even the next shape is going to be rather tedious.)

Is it not remarkable how such a simple rule can give rise to so pretty a shape?

13.9 Map-in-a-map

A USAMO problem Our last example of iterations in geometry will be based on the following elegant problem which appeared in the USA mathematical olympiad (USAMO) of 1978.

> *We are given two square maps of the same country, drawn to different scales. The smaller map is placed on and within the boundary of the larger one. Show that there is precisely one point on the small map which represents the same place of the country as the point of the large map that lies directly below it. Give a way of locating this point.*

The use of the word "square" may suggest that the result holds only for maps which are square in shape, but this is clearly no restriction, for any map can be embedded in a larger map that is square in shape.

Notation

Let ABCD denote the larger map, and A′B′C′D′ the smaller one. We use the prime symbol (′) to show the correspondence between the maps: if X is any point on the large map, then the point on the small map corresponding to X will be denoted by X′. We also write X′ = f(X), where f is the function describing how the two maps correspond to one another (in Figure 13.32, A′ = f(A),B′ = f(B), and so on).

Next, we denote by r the scale factor of the small map relative to the large map ($0 < r < 1$). The scaling applies uniformly to all parts of the map, so if two distinct points P and Q of the large map correspond to (distinct) points P′ and Q′ of the small map, then $P'Q' = r \cdot PQ < PQ$. So if "dist" represents the distance function, with dist (P,Q) = the distance between the points P and Q, then the function f has the property that

$$\text{dist}\,(f(P),f(Q)) = r \cdot \text{dist}\,(P,Q) < \text{dist}\,(P,Q),$$

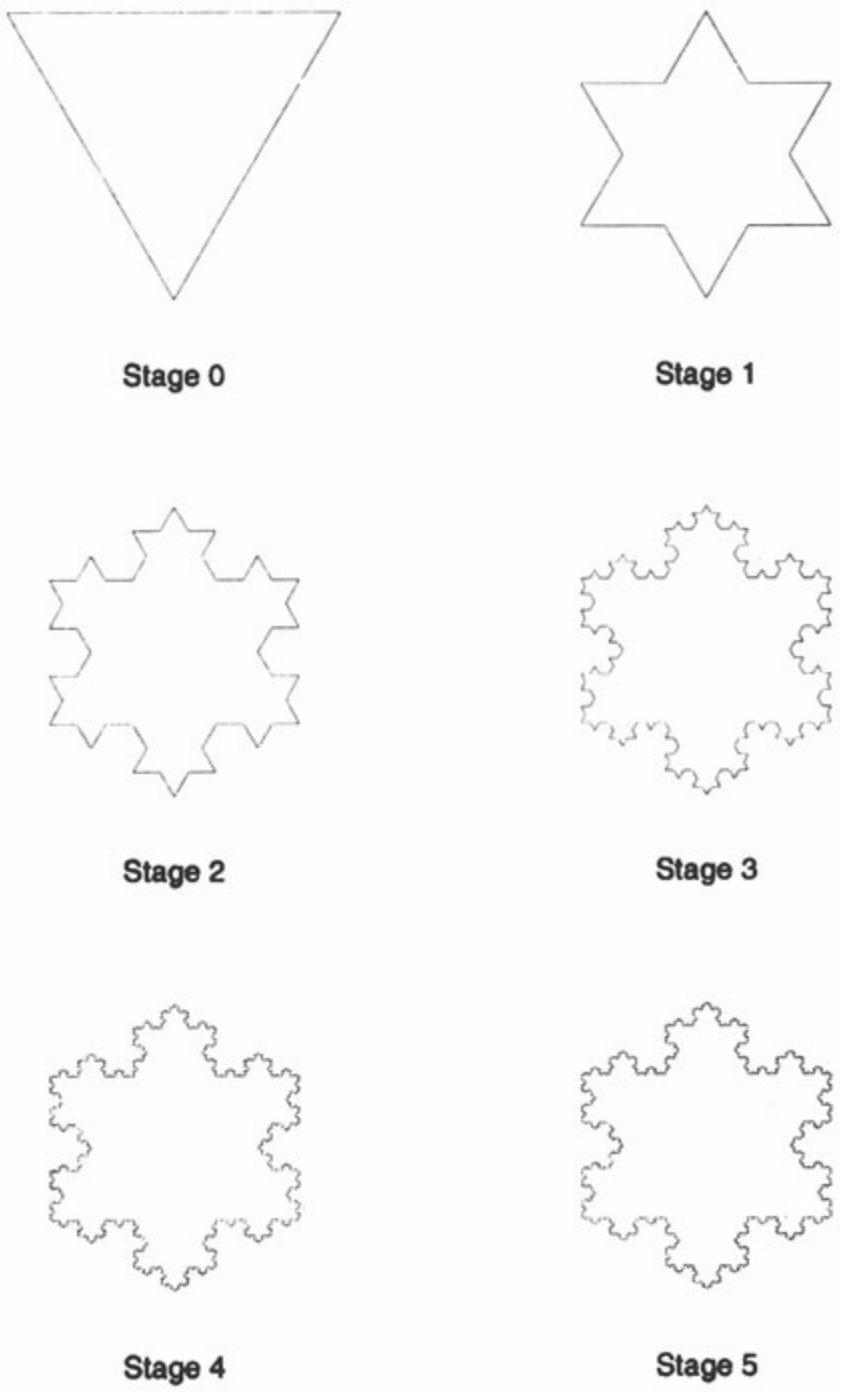


Figure 13.31. *Stages* 0 *to* 5 *of the snowflake curve*

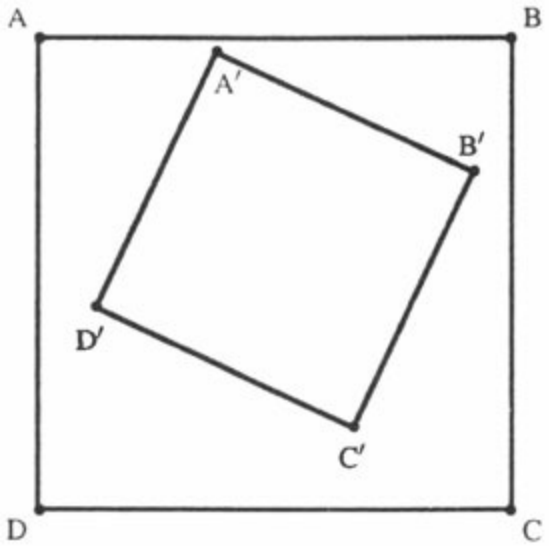


Figure 13.32. *Map-in-a-map*

for all points P≠Q of the large map. Note in particular that for all points P≠Q of the large map, the distance between f(P) and f(Q) is strictly less than the distance between P and Q.

The problem asks us to show that there is one and only one point O of the large map such that f(O) = O (i.e., f has one and only one fixed point), and to show how the point can be located geometrically.

Solution to the problem

Observe that the 'only one' pan is easy: there certainly cannot be *more than one* such point. For if P and Q were both fixed points (P≠Q), then we would have P = f(P) and Q = f(Q), i.e., P′ = P and Q′ = Q, implying that P′Q′ = PQ; but this contradicts the relation P′Q′ < PQ obtained above. So there can be at most one fixed point.

When f is applied to the entire square M = ABCD, we get the smaller square M′. What happens if we apply f to M′? We obviously get a square M″ which lies within M′ and bears exactly the same relation to M′ as M′ does to M. The scale of M″ is r times that of M′, and therefore r2 times that of M. If we are given only M and M′, we can easily construct M″ using Euclidean tools. For, point A″ must lie inside M′ and be located such that △A″A′B′~△A′AB. We can easily do this using the tools of Euclidean geometry.

This can clearly be continued indefinitely: we can apply f to M″ and get M ‴, and so on. The squares thus obtained shrink at a geometric rate, and each lies within the preceding square. So we get a "nested" sequence of squares (see Figure 13.33).

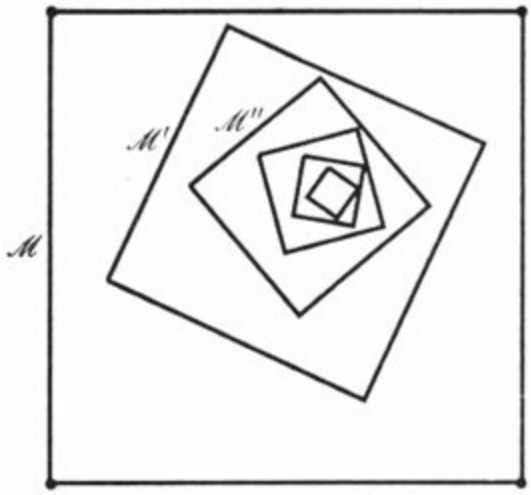


Figure 13.33. *Nested maps*

If we focus attention (see Figure 13.34) on the f-iterates of the vertex A alone, we see that the sequence of iterates

$$A, A', A'', A''', A'''', \ldots$$

quite visibly converges to a limiting point *O. This is the fixed point of* f.

Locating the fixed point geometrically

It is possible to find the fixed point O without too much work. Indeed, we need only two iterates, A′ and A″. (If we are not given A″, then we first construct it as indicated above.) We give below a geometric procedure for locating O.

Observe firstly that for any three points P,Q and R, the image of $\triangle PQR$ under f, namely $\triangle P'Q'R'$, is similar to $\triangle PQR$. Since O is a fixed point (O′ = O), we infer that $\triangle OAA'$ is similar to $\triangle OA'A''$, and $\angle OA'A'' = \angle OAA'$. Therefore A′A″ is tangent (at A′) to the circumcircle of $\triangle OAA'$. This circle can readily be drawn, even though we do not know where 0 lies; for its center must lie on the line through A′ perpendicular to A′A″, and it must also lie on the perpendicular bisector of segment AA′.

Next, we have $\angle OA'A = \angle OA''A'$, therefore AA' is tangent (at A') to the circumcircle of $\triangle OA''A'$. This circle too can be drawn readily, even though we have not yet found *O*, by applying the same logic: its center must lie on the line through A' perpendicular to AA' and also on the perpendicular bisector of segment $A'A''$.

The two circles meet at A' and O, therefore O is located; see Figure 13.35.

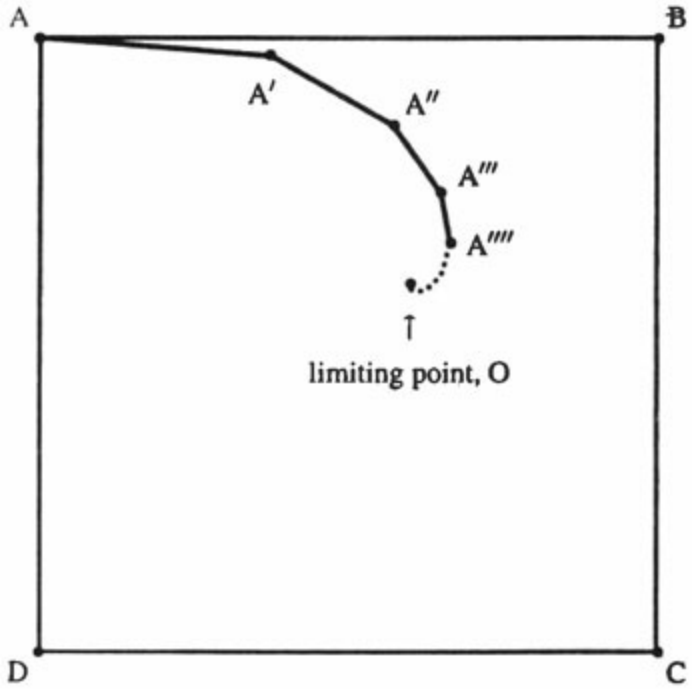


Figure 13.34. *Successive iterates of A*

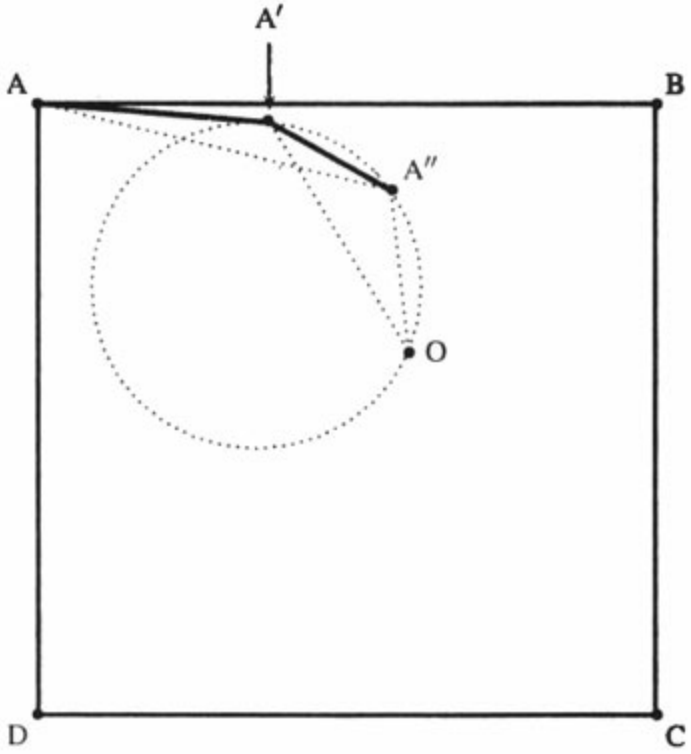


Figure 13.35. *Locating the fixed point* A

Closing remarks

The subject of this section is related to what is known as *fixed point theory*, a very active area of mathematics today—it has whole journals and websites devoted to it! Fixed point theory is part of a subject called *topology*.

The map-in-a-map result is a special case of an extremely important result in this field called Brouwer's *fixed point theorem*. The theorem has all kinds of nice applications in subjects that would seem very far removed from mathematics; for example, economics. The theorem was discovered by a Dutch mathematician, L E J Brouwer (1881–1966).

The "map" result is an application of Brouwer's theorem in *two* dimensions. Even the one dimensional case of the theorem is elegant:

> *Let* f *be a continuous map from a finite interval I into itself; then* f *has a fixed point*.

The two dimensional version of the theorem states the following:

> *Let* f *be a continuous function from a convex, bounded set* S *into itself*;

then fhos *a fixed point*.

The word "continuous" in these two statements here has a technical meaning which we will not go into. In the context of our "map" result, what it means is that while we are allowed to shrink the map as much as we wish, and place it inside the larger map at any orientation, even upside down, we are not permitted to *tear* it. Tearing the map would qualify as a "discontinuous" operation.

The theorem may be similar extended to cover the case of three dimensions.

Here is a pretty application of the theorem; it is called the *coffee cup theorem*. It says that if you take a cup of coffee and stir the coffee with a spoon, then at the end there is some coffee particle that is in exactly the same place that it was before the stirring. If you "push" this particle out of its position, then somewhere else in the cup a particle will end up in its original position!

Exercises

13.9.1 Consider the case when the sides of the small map are parallel respectively to the sides of the large map. Two subcases of interest are: (a) $A'B' \parallel AB$, $B'C' \parallel BC$; (b) $A'B' \perp AB$,$B'C' \perp BC$.

Find simpler ways of locating the fixed point O in these two subcases.

13.9.2 What happens if the small map juts outside the large map? Does a fixed point necessarily exist?

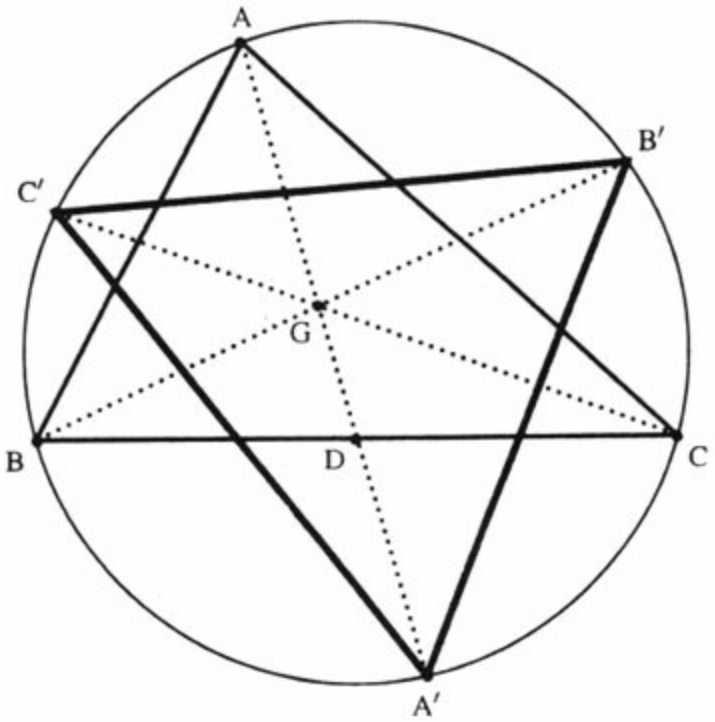


Figure 13.36.

13.9.3 I take a walk as follows: from my house I walk 1 km East, then 1/2 km North, then 1/4 km West, then 1/8 km South, then 1/16 km East, then 1/32 km North, and so on. Where do I find myself eventually? (Give a geometric argument for locating the limiting position.)

13.10 Appendix—analysis of the medians-iteration

We now take up the analysis of the medians iteration studied in Section 13.4. We had said then that the analysis was rather challenging. We now give the full details. Our approach will be.to translate the geometric iteration into an algebraic one, and then recast it into a geometric context using the idea of **BARYCENTRIC COORDINATES**. (For those who are unfamiliar with this term, a full definition is given below.)

★ ★

In Figure 13.36, AD,BE,CF are the medians of triangle ABC, and their lengths are ma,mb,mc, respectively. The medians, when extended, meet the circumcircle Γ at points A',B',C'. Let the sides of triangle ABC be a,b,c, and let the lengths of the medians be ma,mb,mc, respectively. Then $a' : b' : c' =$

ama : bmb : cmc (this was proved earlier). It follows that

$$a'^2 : b'^2 : c'^2 = a^2(2b^2 + 2c^2 - a^2) : b^2(2c^2 + 2a^2 - b^2) : c^2(2a^2 + 2b^2 - c^2).$$

Let $x : y : z = a^2 : b^2 : c^2$ and $x' : y' : z' = a'^2 : b'^2 : c'^2$. Then, we have:

$$x' : y' : z' = x(2y + 2z - x) : y(2z + 2x - y) : z(2x + 2y - z).$$

Noting that the sum of the numbers in the triple on the right side is:

$$x(2y+2z-x)+y(2z+2x-y)+z(2x+2y-z) = 4(xy+yz+z\kappa)-(x^2+y^2+z^2),$$

we define a mapping on triples of real numbers whose sum is 1, as follows. If (x,y,z) is such a triple, i.e., with $x + y + z = 1$, then its image is $(x',y',z') = f(x,y,z)$, where

$$x' = x(2y + 2z - x)\ 4(xy + yz + zx) - (x^2 + y^2 + z^2),\ y' = x(2z + 2x - y)\ 4(xy + yz + zx) - (x^2 + y^2 + z^2),\ z' = x(2x + 2y - z)\ 4(xy + yz + zx) - (x^2 + y^2 + z^2), \quad (13.1)$$

We need to show that the iterates tend to the triple (1/3, 1/3, 1/3).

Transformation of problem, using barycentric coordinates

Let us regard (x,y,z) as the *barycentric coordinates* of a point P with reference to some fixed equilateral triangle T. Before going on, we define what this means.

Oriented area Given a polygon P with its boundary (usually denoted by ∂P) traversed in an anticlockwise or clockwise direction, its *oriented area* (denoted by [P]) is its ordinary area affixed with a *positive* sign if the direction is anticlockwise, and a *negative* sign if the direction is clockwise.

From the definition, it follows that [ABC] = -[ACB] for any triangle ABC, for the "sense" of the closed path A → B → C → A is opposite to that of the path A → C → B→ A.

Barycentric coordinates Given a fixed triangle T with vertices A,B,C (we shall call it as the *triangle of reference*), for any point P in the plane of the triangle, let

$$x : y : z = [PBC] : [PCA] : [PAB\ ].$$

We shall assume that the labelling of triangle ABC is done in an anticlockwise

order (so [ABC] is positive). The location of P is uniquely determined once the ratios x : y : z are known (this follows from elementary geometry). We call the triple (x,y,z), or the ratios x : y : z, the *barycentric coordinates* of P. Examples: The barycentric coordinates of vertex A are (1,0,0), those of the mid-point of side BC are (1/2, 1/2, 0) , and those of the centroid of the triangle are (1/3, 1/3, 1/3). The only triple that does not correspond to any point in the plane is (0,0,0).

Note that the triple (kx,ky,kz) corresponds to the same point as (x,y,z). (So the centroid may also be described by the triple (1, 1, 1), and the mid-point of side BC by the triple (1, 1, 0) .) If x,y,z all have the same sign, then the point P lies inside the triangle, and if x < 0,y > 0,z > 0, then P lies within the angle determined by the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ but on the opposite side of BC as compared with A.

★ ★

Coming back to our iteration, we see that the image of a point P, determined by a triple (x,y,z), is a point P′, determined by the triple (x′,y′,z′), where x′,y′,z′ are computed using equation (13.1). We need to check whether the iterates of P under this mapping tend to the centroid G of the triangle of reference, T.

We assign the following coordinates to the vertices of T = △UV W:

$$U = (0,3), V = (-1,0), W = (1,0).$$

Then. the centroid of T is G = (0,1⁄3), and the coordinates of the point P corresponding to the triple (x,y,z) are

$$x \cdot U + y \cdot V + z \cdot W \; x + y + z = z - y \; x + y + z, \; x3 \; x + y + z$$

We first establish two preliminary results (“lemmas”).

Lemma 1. *The equation of the incircle of* T *is* x2 + y2 + z2 = 2(xy + yz + zx). *Points within the incircle have* x2 + y2 + z2 < 2(xy + yz + zx); *points outside the incircle have* x2 + y2 + z2 > 2(xy + yz + zx).

For, the square of the distance between P and G is

$$PG2 = z - y \; x + y + z2 + x3 \; x + y + z - 1 \; 32$$

$$= 4(x2 + y2 + z2 - xy - yz - zx)\ 3(x + y + z)2 .$$

Since the radius of the incircle is 1⁄3, we compute the quantity PG2 - 1⁄3:

$$PG2 -1\ 3 = 4(x2 + y2 + z2 - xy - yz - zx)\ 3(x + y + z)2 -1\ 3$$

$$= x2 + y2 + z2 - 2(xy + yz + zx)\ (x + y + z)2 .$$

So x2 + P + z2 - 2(xy + yz + zx) is negative, zero or positive, according to whether P is within, on, or outside the incircle of the triangle. ❑

Lemma 2. *The equation of the circumcircle of* T *is* xy + yz + zx = 0. *Points within the circumcircle have* xy + yz + zx > 0, *and points outside the circumcircle have* xy + yz + zx < 0.

Since the radius of the circumcircle of the triangle is 2⁄3, we compute the quantity PG2 - 4⁄3, using the expression for PG2 from above. After simplification, we get:

$$PG2 -4\ 3 = 4(x2 + y2 + z2 - xy - yz - zx)\ 3(x + y + z)2 -4\ 3 = -4(xy + yz + zx)\ (x + y + z)2 .$$

Therefore, xy + yz + zx is positive, zero or negative, according to whether P is within, on, or outside the circumcircle. ❑

Theorem 13.1 *If P lies within the incircle, the image point P′ lies closer to the center of the incircle than P, i. e ., GP′ < GP. If P lies on the incircle, then so does P′; i. e the distance from G stays fixed. If P lies outside the incircle, then P′ too lies outside the incircle. If P lies on the circumcircle, then so does P′.*

Proof. The coordinates of the image point P′, with x′,y′,z′ given by equations (13.1), are

$$(y - z) \cdot (2x - y - z)\ 4(xy + yz + zx) - (x2 + y2 + z2)\ ,\ x \cdot (x - 2y + 2z)3\ 4(xy + yz + zx) - (x2 + y2 + z2).$$

The square of the distance between P′ and G(i.e.,P′G2) is therefore given by

$$(y - z) \cdot (2x - y - z)\ 4(xy + yz + zx) - (x2 + y2 + z2)2 + 1\ 3 \cdot 2x2 - y2 - z2 - 2xy - 2xz + 4yz\ 4(xy + yz + zx) - (x2 + y2 + z2)2.$$

This simplifies (after some effort!) to

$$P'G2 = 4\,3 \cdot (x2 + y2 + z2 - xy - yz - zx)2\,[4(xy + yz + zx) - (x2 + y2 + z2)]2.$$

Therefore,

$$P'G2\ -1\ 3 = (x + y + z)2 \cdot (x2 + y2 + +z2 - 2xy - 2yz - 2zx)\,[4(xy + yz + zx) - (x2 + y2 + z2)]2\ . \qquad (13.2)$$

Observe that if x;y : z = 1 : 1 : 1, then P′G = 0, i.e., P′ = G. Further , if P≠G, then

$$P'G2\ PG2 - 1 = 9 \cdot (x2 + y2 + z2 - 2xy - 2yz - 2zx) \cdot (xy + yz + zx)\,[4(xy + yz + zx) - (x2 + y2 + z2)]2\ . \qquad (13.3)$$

From equations (13.2) and (13.3), we see that the following possibilities can arise.

- If P lies within the incircle, then we have

  $$xy + yz + zx > 0, x2 + y2 + z2 < 2(xy + yz + zx),$$

  so P′G < PG. Therefore, points within the incircle are pulled *towards* the center of the triangle.
- If P lies on the incircle, then

  $$x2 + y2 + z2 = 2(xy + yz + zx),$$

  giving P′G2 - 1⁄3 = 0; therefore, P′ too lies on the incircle.
- If P lies outside the incircle, i.e.,

  $$x2 + y2 + z2 > 2(xy + yz + zx),$$

  then P′G2 - 1⁄3 > 0, which means that P′ too lies outside the incircle.
- If P lies outside the in circle but within the circumcircle, then

  $$xy + yz + zx > 0, x2 + y2 + z2 > 2(xy + yz + zx),$$

  so P′G > PG. Therefore, points in this region are pushed *away* from the center of the triangle.
- If P lies on the circumcircle, then P′G = PG, therefore, P′ too lies on the circumcircle.
- If P lies outside the circumcircle, then P′G < PG. So such points are once

again pulled *towards* the center of the triangle.

Convergence of the iteration. Let the quantity t be defined by

$$t = x2 + y2 + z2\ xy + yz + zx. \tag{13.4}$$

Note that we cannot have $t < 1$, as x,y,z are real. We now write the relation

$$P'G2\ PG2 = (x + y + z)2 \cdot (x2 + y2 + z2 - xy - yz - zx)\ [4(xy + yz + zx) - (x2 + y2 + z2)]2 \tag{13.5}$$

in terms of t. Since

$$(x + y + z)2 = (xy + yz + zx) \cdot (t + 2),$$

$$x2 + y2 + z2 - xy - yz - zx = (xy + yz + zx) \cdot (t - 1),$$

$$4(xy + yz + zx) - (x2 + y2 + z2) = (xy + yz + zx) \cdot (4 - t),$$

we get:

$$P'G2\ PG2 = (t + 2) \cdot (t - 1)\ (4 - t)2\ . \tag{13.6}$$

Let $g(t) = (t + 2) \cdot (t - 1)/(4 - t)2$. Then $g(2) = 1$; this confirms what we already know: that points on the incircle stay on the incircle.

A table of values of g is of interest:

t	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
$g(t)$	0	0.0816	0.201	0.375	0.628	1	1.55	2.40	3.75	6	10

We see that as t drops from 2 to 1, g(t) decreases from 1 to 0; moreover, the slope of g(t) at $t = 0$ is 0. This explains why the iterates converge to the equilateral shape so rapidly.

Does the starting point always lie within the incircle? We must check that at the start of the iteration, the point P does lie inside the incircle. But this is easy. Recall that at the start we had

$$x : y : z = a2 : b2 : c2.$$

Therefore, to show that $x2 + y2 + z2 < 2(xy + yz + zx)$ we must show that

$$a4 + b4 + c4 < 2(a2b2 + b2c2 + c2a2).$$

In other words, we must show that

$$a4 + b4 + c4 - 2(a2b2 + b2c2 + c2a2) < 0.$$

Rather conveniently for us, the expression on the side factorizes exactly, as

$$-(a + b + c) \cdot (a + b - c) \cdot (b + c - a) \cdot (c + a - b).$$

In this factorization, the quantities a + b - c,b + c - a and c + a - b are all positive (strictly), by virtue of the triangle inequality ("any two sides of a triangle are together greater than the third side"); also, trivially, a + b + c is positive. It follows that the quantity

$$a4 + b4 + c4 - 2(a2b2 + b2c2 + c2a2)$$

is negative, and therefore, the inequality $a4 + b4 + c4 < 2(a2b2 + b2c2 + c2a2)$ is true.

This means that the triple (x,y,z) with which we start the iteration satisfies the relation

$$x2 + y2 + z2 < 2(xy + yz + zx).$$

So, at the start of the iteration, the point P lies strictly within the incircle, and the iteration pulls if towards the center of the triangle, at an ever increasing rate. , ❑

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

We may anticipate from the above that the triangles for which convergence is slow are those for which the ratio $(a2 + b2 + c2)/(ab + bc + ca)$ is close to 2; the closer the ratio is to 2, the slower the convergence. This will be the case for an isosceles triangle whose base is small in relation to the equal sides; for example, a triangle with sides a = 100 = b and c = 1, with ratio 20001⁄10200 ≈ 1.961.

Let us try the iteration using this triangle. The initial values of x,y,z are given by:

$$X = 1002\ 1002 + 1002 + 12 \approx 0.499975, y \approx 0.499975, z \approx 0.00005.$$

Starting with this triple, the first few iterates are the following:

(0.49998, 0.49998, 0.00005), (0.49990, 0.49990, 0.00020),

(0.49960, 0.49960, 0.00080), (0.49841, 0.49841, 0.00317),
(0.49380, 0.49380, 0.01240), (0.47731, 0.47731, 0.04538),
(0.43254, 0.43254, 0.13492), (0.36922, 0.36922, 0.26157),
(0.33729, 0.337 29, 0.32542), (0. 33338, 0.33338,0.33324),
(0.33333, 0.33333, 0.33333), (0.33333, 0.33333 ,0.33333).

We see that even with such a skewed starting triple, convergence to the equilateral shape is very rapid.

Remark. As noted above, if the point P lies in the annular region between the incircle and the circumcircle of T, then $P'G > PG$; it is "pushed away" from the center of the triangle. And if P lies outside the circumcircle, then $P'G < PG$; it is "pulled back" towards the center of the triangle.

This has the following remarkable consequence: if the starting point lies out-side the incircle of T, then its iterates cannot converge to any position whatever. Instead, they are forced to "wander around" aimlessly outside the incircle; neither can they enter within the circle, nor can they recede very far from it—a strange situation, indeed !

Chapter 14

Techniques of Proof

We outline here a general methodology for investigating iteration problems and for *proving* results. The material is a little more advanced than that in the earlier chapters. We study in detail two iterations discussed earlier: the SSQ iteration, and the four-numbers iteration. We had noticed at the time that for both the iterations, irrespective of which seed is chosen, orbits show predictable behaviour: in the SSQ iteration all orbits seem to converge either to the fixed point 1 or to the 8-cycle

$$\langle 4,16,37,58,89,145,42,20\rangle,$$

and in the four-numbers iteration—still simpler—all orbits seem to converge to the fixed point (0,0,0,0). We remarked at the end of the chapter that experimental observation does *not* make for proof; we cannot be certain that the observed behaviour will hold in each and every instance. What we shall study now is a way of "being certain".

14.1 The SSQ iteration

Recall that if n is a positive integer, then SSQ(*n*) denotes the sum of the squares of the digits of n (when n is written in base-10); e.g., SSQ(35) = 32 + 52 = 34. Let n be a number with k digits; then, clearly,

$$\text{SSQ}\,(n) \leq 81k,$$

because the largest SSQ values occur for numbers that consist solely 9's. Note that equality holds precisely when n = 10k - 1. We write this assertion compactly as follows:

$$\text{If } n < 10k, \text{ then SSQ}\,(n) \leq 81k.$$

As a corollary we conclude that:

$$\text{If } n < 1010, \text{ then } SSQ(n) \leq 810.$$

Now among the numbers between 1 and 810, the number with the largest SSQ is 799. Since SSQ(799) = 72 + 92 + 92 = 211, we conclude that:

$$\text{If } n < 1010 \text{ then } SSQ(SSQ(n)) \leq 211.$$

The notation SSQ(SSQ(n)) looks clumsy, so we use a superscript:

- we use $SSQ^{(2)}(n)$ to stand for SSQ(SSQ(*n*));
- we use $SSQ^{(3)}(n)$ to stand for SSQ (SSQ (SSQ(n)));

and so on—the superscript in brackets indicates the number of times that the SSQ operation is applied. So we have:

$$\text{If } n < 1010 \text{ then } SSQ(2)(n) \leq 211.$$

Continuing, we note that among the numbers between 1 and 211, the number with largest SSQ is 199, with SSQ (199) = 12 + 92 + 92 = 163, therefore:

$$\text{If } n < 1010 \text{ then } SSQ(3)(n) \leq 163.$$

One last round of such reasoning is needed: among the numbers between 1 and 163, the largest SSQ occurs for 99, with SSQ(99) = 92 + 92 = 162. Therefore:

$$\text{If } n < 1010 \text{ then } SSQ(4)(n) \leq 162.$$

Since 162 is greater than 99, we cannot guarantee any more reduction in the SSQ value by further iterations; the "ceiling" continues to remain at 162. For convenience, we let *S* denote the set of numbers {1,2,3,…,161,162}.

We may now confidently assert the following.

a. If $n < 1010$, then after at most four iterations we reach a number in the set S.
b. If $1 \leq n \leq 162$, then $1 \leq SSQ(n) \leq 162$; i.e., if $n \in S$, then $SSQ(n) \in S$. In other words, once we reach a number in the set S, we stay within this set.

These observations provide the lead for the next step. The point to note is that this reasoning applies to any initial input, no matter how large. When the input number n is very large, SSQ (n) will be correspondingly small in comparison with n. For instance:

$$\text{If } n < 101000 \text{ then } SSQ(n) \leq 81000.$$

Since 81000 < 1010, the argument used earlier shows that if n < 101000, then *in at most five iterations* we reach a number in the set S.

The following general conclusion now becomes unavoidable:

> *For any choice of input, no matter how large, repeated application of SSQ eventually produces a number in the set* S.

We also know (see item (b), above) that once we reach a number in the set S, application of SSQ does not take us out of the set; so the conclusion may be stated in a stronger form:

> *For any choice of input, no matter how large, repeated application of* SSQ *eventually traps us inside the set* S.

Now this set is finite: it has just 162 elements. As we apply SSQ over and over again, starting at some number in S, it must happen sooner or later that we encounter a number reached earlier. From this point on the sequence must repeat, and this means that we have reached a cycle! So:

> *No matter what input we start with, we must eventually reach a cycle.*

Note that this argument yields no information on the length or composition of the cycle, or a bout the number of cycles. Such information can be obtained only after a more detailed analysis. This, however, is easy to do. We start with the element 1 and trace its orbit; in this case the orbit consists of just the number 1 itself. (We stop the computation once the orbit closes.) Next, we select a number in S which does not lie in ⟨1; SSQ ⟩ , say the element 2, and compute its orbit, ⟨2; SSQ ⟩ :

$$\langle 2,4,16,37,58,89,\ldots\rangle.$$

Once again we select a number in S that does not appear in any of the orbits computed till now (say the number 3), and trace *its* orbit, ⟨ 3; SSQ ⟩ :

$$\langle 3,9,81,65,61,\ldots\rangle.$$

Noting that 5 has not appeared in any orbit computed till now, we now compute the orbit $\langle 5;\ SSQ\ \rangle$; and so on; we proceed systematically in this manner. It should be clear that when the integers between 1 and 162 have all been accounted for, *all* the cycles of the SSQ function will have been found. The task may appear a bit tedious—but not if we have access to a computer!

We shall not pursue the analysis further at this point; you are invited to do so.

General comments

The proof technique used above is quite general and applies to many different types of iterations. Let f denote the function being iterated. In essence, the technique requires that we identify a finite (and hopefully small) set *S* such that the following features hold:

a. regardless of where exactly we start, the output will at some stage belong to S; that is, for any x, it will happen that $f(k)(x) \in S$ for some finite number k; and

b. when the input belongs to S, then so does the output (i.e., if $x \in S$, then $f(x) \in S$) .

If these conditions maintain, then we are assured of reaching a cycle, no matter where we start the iteration.

"BIack holes"

Let us change the metaphor in describing the above features. In essence we seem to be looking for a set S that exerts a strong "gravitational pull" on orbits; they get drawn towards and into it, irresistibly. If an orbit "falls" into *S*, then there is no question of its ever "getting out" again. Do these phrases remind us of certain well known objects in astronomy? Surely, yes: they remind us black holes! In their honor, then, if for a particular iteration f we find a set S with the features just described, we shall call it a "black hole set." A more accepted (and more somber!) term for such a set is *basin of attraction*.

14.2 The SCB and SFT iterations

A somewhat different approach to the listing of cycles was used in the section titled "Two-digit iteration", in Chapter 11. It can be used with profit in many instances, especially when the sets involved are provably finite, and good computing facilities are at hand. Here is an application of this technique. Rather than consider the SSQ function which we have just discussed in detail, we consider two related functions:

- the SCB function, where SCB (x) is the sum of the cubes of the digits of x when written in base-l0, e.g., SCB(123) = 13 + 23 + 33 = 36; and
- the SFT function, where SFT(x) is the sum of the fourth powers of the digits of x when written in base-10, e.g., SFT(123) = 14 + 24 + 34 = 98.

Our objective will be to find all the cycles of SCB and SFT.

Finding black hole sets for SCB and SFT

We start by finding values of M and N such that

- if $1 \le x \le M$, then $1 \le SCB(x) \le M$, and if $x > M$, then $SCB(x) < x$;
- if $1 \le x \le N$, then $1 \le SFT(x) \le N$, and if $x > N$, then $SFT(x) < x$.

Possible values are M = 3000 and N = 30000, because:

- for numbers below 3000, the largest SCB value is achieved for 2999, with SCB(2999) = 2195 < 3000; and further, if x > 3000, then SCB(*x*) < x;
- for numbers below 30000, the largest SFT value is achieved for 29999, with SFT(29999) = 26260 < 30000; and further, if x > 30000, then SFT (x) < x.

It follows that:

- for the function SCB, the set S1 = {1,2,3,…,3000} is suitable; and
- for the function SFT, the set S2 = {1,2,3,…, 30000 } is suitable.

Extending the domains of SCB and SFT

Next, we allow SCB and SFT to operate on *sets* rather than just single numbers; thus we allow operations such as

$$\text{SCB}(\{15,20,25,30\}) = \{126,8,133,27\},$$

$$\text{SFT}(\{15,20,25,30\}) = \{626,16,641,81\}.$$

We also define new functions FCB and FFT which (like SCB and SFT) operate on set as follows: if X is a set of whole numbers, then

- $\text{FCB}(X) = X \cap \text{SCB}(X)$, and
- $\text{FFT}(X) = X \cap \text{SFT}(X)$.

For example, if X = {1,2,3,4,…, 18,19,20}, then

$$\text{FCB}(X) = \{1,2,8,9\}, \quad \text{FFT}(X) = \{1,2,16,17\}.$$

Listing the cycles of SCB

The last step is simplicity itself: we compute the orbit ⟨S1; SCB ⟩ . That is, we apply FCB to S_1 iteratively till we get an "FCB-invariant set"—a set which remains the same under action by FCB. (This is where heavy computing machinery is required! The author used *MATHEMATICA* to do the work.) After 20 iterations of FCB on S1 we get the following set X:

$$X = \{1,55,133,136,153,160,217,244,250,352,370,371,407,919,1459\},$$

and this is FCB-invariant. Applying SCB to X gives back the same set, but with the elements listed in another order:

$$\{1,250,55,244,153,217,352,136,133,160,370,371,407,1459,919\}.$$

The cycles of SCB are recovered from the two lists simply by tracing the orbits of the various numbers in X; by doing so, we get the following cycles:

1-cycle	⟨1⟩, ⟨153⟩, ⟨370⟩, ⟨371⟩, ⟨407⟩
2-cycle	⟨136, 244⟩, ⟨919, 1459⟩
3-cycle	⟨55, 250, 133⟩, ⟨160, 217, 352⟩

These, then, are *all* the cycles of SCB; there are no cycles of order greater than 3.

Listing the cycles of SFT

We do exactly the same for SFT: we compute the orbit ⟨S2; FFT ⟩. That is, we apply SFT to S2 iteratively till we get an "FFT-invariant set"—a set which remains the same under action by FFT. Using *MATHEMATICA* we do 60 iterations of SFT on S2, and get the following set Y :

Y = {1,1138,1634,2178,4179,4338,4514,6514,6725,8208,9219,9474,13139},

and this is FFT-invariant. Applying SFT to Y gives back the same set, but with the elements listed in a different order:

{1,4179,1634,6514,9219,4514,1138,2178,4338,8208,13139,9474,6725}.

It is now an easy exercise to trace out the cycles of SFT; we get the following list.

1-cycle	⟨1⟩, ⟨1634⟩, ⟨8208⟩, ⟨9474⟩
2-cycle	⟨2178, 6514⟩
7-cycle	⟨1138, 4179, 9219, 13139, 6725, 4338, 4514⟩

These, then, are *all* the cycles of SFT. (The 7-cycle certainly comes as a surprise!)

Exercises

14.2.1 Consider the technique described above, which we have used to list the cycles of SCB and SFT. What makes it work?

14.2.2 Do these ideas throw fresh light on the topic of self referential numbers? Can a black hole set S be identified for this iteration?

14.2.3 Explain why the card trick described at the start of the book (in Chapter 1) "works". (You must explain why the chosen card invariably ends up exactly in the middle of the pack after four iterations.)

14.2.4 Apply, the techniques described in this chapter to find the cycles for the following iterations (use suitable computing facilities; the more powerful the better!):

- f1(n) = sum of the fifth powers of the digits of n when it is written in base-10, e.g., f1(123) = 15 + 25 + 35 = 276.
- If n = (abc…)10, written in base-10, then

$$f2(n) = (a + 1) \times (b + 1) \times (c + 1) \times \cdots.$$

For example, $f2(123) = 2 \times 3 \times 4 = 24$.

It turns out that f2 has precisely two cycles—a 1-cycle and a 9-cycle.

14.3 The four-numbers iteration

Another theme used in the analysis of iterations is illustrated by the four-numbers game where the iterative function is

$$(a,b,c,d) \mapsto (|a - b|,|b - c|,|c - d|,|d - a|).$$

Define the *life* of a quadruple to be the number of steps it takes to reach (0,0,0,0).

Example 1. The life of (3,4,6,1) is 4, because the trajectory followed by this seed is:

$$(3,4,6,1) \mapsto (1,2,5,2) \mapsto (1,3,3,1) \mapsto (2,2,2,2) \mapsto (0,0,0,0) \mapsto \cdots.$$

If (0,0,0,0) cannot be reached from some quadruple, then the life of that quadruple is considered to be infinite. Our task is therefore to show that every quadruple of integers has finite life.

We start by noting that for any four numbers a,b,c,d, the life of (2a,2b,2c,2d) is the same as that of (a,b,c,d). This is so in the orbit of (2a,2b,2c,2d), every entry is double the corresponding entry in the orbit of (a,b,c,d), so (0,0,0,0) is reached (if at all) after the same number of steps. More generally, the life of (ka,kb,kc,kd) is the same as that of (a,b,c,d), for any positive integer k. To convince yourself of this, compare the orbits of (3,4,6,1) and (9,12,18,3) (the second quadruple is "3 times" the first one.)

Next, define the *height* of a quadruple of numbers to be its largest number. For example.

$$\text{height } (3, 4, 6, 1\ ) = 6.\ \text{height } (13,6,19,40) = 40.$$

The height of (0,0,0,0) is 0. and this is clearly the only quadruple with zero height. (We consider only quadruples of non-negative integers.)

Exercises

14.3.1 Show that the height of the offspring does not exceed the height of the parent. In other words, show that if we have $(a, b, c, d) \to (e, f, g, h)$ then

$$\text{height } (e, f, g, h) \leq \text{height } (a, b, c, d).$$

14.3.2 Let any quadruple of integers be chosen as seed. Show that in *at most four steps* we reach a quadruple in which the numbers are all even.

Example 2. Let the seed be (25,12,10,40) : Its orbit is:

$$(25,12,10,40)\mapsto(13,2,30,15)(11,28,15,2)$$

$$\mapsto(17,13,13,9)\mapsto(4,0,4,8)\mapsto\cdots,$$

so we have reached an all-even-numbers quadruple in four steps. What you must show is that more than four steps are never needed, no matter what quadruple we start with.

We now make use of observations made earlier. Suppose that staring with the quadruple (a,b,c,d), where a,b,c,d are positive integers, within four steps or less we reach the quadruple $(2e, 2f, 2g, 2h)$ whose entries are all even. We have already noted that the life of $(2e, 2f, 2g, 2h)$ is the same as that of (e, f, g, h). What relationship holds between the life of (a, b, c, d) and that of (e, f, g, h).?

Exercise

14.3.3 Suppose that (a,b,c,d) reaches $(2e, 2f, 2g, 2h)$ in four (or fewer) steps. Show that the height of (e, f, g, h) is at most *half* the height of (a,b,c,d).

Since our interest is only in the *life* of the quadruple, once we reach a quadruple whose entries are all even, no essential difference is brought about if we halve all its entries; that is, if we replace $(2e, 2f, 2g, 2h)$ by (e, f, g, h). The exercise therefore tells us that in four steps at most, we achieve a *halving* (*or better) of the height*.

Exercise

14.3.4 Show how these observations enable us to conclude that every quadruple of integers eventually reaches (0,0,0,0).

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

[You could argue in the following manner. Suppose that the height of the starting quadruple is less than 1000. Then in at most four steps, we reach a quadruple with height less than 500, and in at most four more steps, one with height less than 250. Continuing this line of argument, we see that in at most forty steps we reach a quadruple with height less than 1. This can happen only when all the entries are 0, so we have reached our desired destination. What role has the number 1000 played in this argument? How should the argument be framed if the numbers in the starting quadruple are 5-digit numbers?]

Alternative treatment

Another form of presenting the argument is the following. We define a function f(x) for positive values of x as follows. Consider the quadruples whose entries are non-negative integers less than x. From amongst these, select the one with the *longest* life; its life is then taken to be the value of f(x). (Note that x itself does not have to be an integer.) For example:

- f(1) = 0; for if each number of the quadruple is a non-negative integer less than 1, the quadruple must be (0,0,0,0), whose life is 0.
- f(2) = 4, because within the set of quadruples with largest entry less than 2, the "worst one" (the one with longest life) is (1,0,0,0), whose life is 4.

Exercises

14.3.5 Compute the values of f(0.9),f(1.5),f(3) and f(4).

14.3.6 Show that if x and y are positive numbers with $x \leq y$ then $f(x) \leq f(y)$.

14.3.7 Show that if $x < 1$ then $f(x) = 0$.

14.3.8 Show that to prove that every quadruple reaches the fixed point (0,0,0,0) at some stage, it is enough to prove that f(x) has a finite value for every positive number x.

14.3.9 Show that $f(x) \leq f\ x\ 2 + 4$ for any $x > 0$.

14.3.10 Use the inequality proved above to show that f(x) is finite for every positive number x.

14.3.11 From an earlier chapter you know that if we worked with triples rather than quadruples, the behaviour of the iteration is quite different—we do not always reach the fixed point (0,0,0). Where does the analysis outlined above fail when it is applied to triples? The operation in this case is:

$$(a,b,c) \mapsto (|a - b|,|b - c|,|c - a|).$$

14.3.12 What happens if we introduce *fractions* into the picture (in other words, if we consider quadruples of fractions rather than only integers)? Here is an example with fractions:

$$2\ 3,\ 3\ 7,\ 5\ 12,\ 3\ 19 \mapsto 5\ 21,\ 1\ 84,\ 59\ 228,\ 29\ 57.$$

Does every quadruple of fractions have finite life?

14.3.13 * Can the above analysis be extended to quadruples of *real numbers*? Prove that it cannot; that is, show that there exist real numbers a,b,c,d for which the quadruple (a,b,c,d) does not have finite life.

14.3.14 Consider once again the problem of explaining why decimals recur: i.e., of explaining why for any two positive integers a,b, the decimal obtained from the division $a \div b$ necessarily repeats. (The "terminating" case may be regarded as one where the repeating portion has a length of 1; e.g., in the case of $43 \div 25 = 1.7200\ldots$ where the 0 repeats indefinitely. Also, as the same example shows, the

repeating portion need not start right after the decimal point).

Do the ideas described in this chapter throw fresh light on this problem?

Concluding comments

It should be clear by now that there is no single "standard" technique for proving assertions about iterations; contrast the different styles of reasoning used for the SSQ iteration and the four-numbers iteration. It appears that one has to mould the approach very finely to the particularities of the iteration being studied.

Chapter 15

Solving Equations

You may have wondered about how the study of iterations began at all. An early encounter with iterations comes in the area of *solving of equations*. This is the subject of the present chapter.

Linear and quadratic equations in one unknown

The idea of a linear equation in one unknown is surely a familiar one, e.g., equations such as $3x + 2 = 8$ in the unknown x. Such equations are easy to solve. For example, the equation $3x + 2 = 8$ is solved as follows: $3x + 2 = 8$, $\therefore$ $3x = 8 - 2 = 6$, $\therefore$ $x = 2$.

Quadratic equations in a single unknown, so-called because of their squared terms, can also be solved in a standard manner. For example, the equation $x2 - 4x + 3 = 0$, in the unknown x, is solved by the standard technique of "completing the square": $x2 - 4x + 3 = 0$, $\therefore$ $x2 - 4x + 4 = 1$ (by "completing the square"), $x - 2 = \pm 1$ (taking square roots), $\therefore$ $x = 1$ or $x = 3$. Please verify that both $x = 1$ and $x = 3$ are solutions of the original equation.

Polynomial equations of higher degree

You may not be familiar with the fact that similar methods (but rather more complicated) are available for solving third degree and fourth degree equations (also known respectively as cubic equations and biquadratic equations or quartic equations); that is, equations such as

$$x3 - 10x + 1 = 0, x4 - x3 - 100x + 7 = 0.$$

These methods were found by the "Italian school," comprising of the trio of

mathematicians Niccolo Tartaglia (1499–1557), Gerolamo Cardano (1501–1576) and Lodovico Ferrar (1522–1565), and date back to the fifteenth century, in fact to the early years of the Renaissance. Though these methods are not too well known at the school level, descriptions can be found in standard texts, so they can be considered as well known. (We add here, in passing, that their biographies make for fascinating reading; see [DUNHAM].)

Other kinds of equations

However not all equations can be solved so simply. Here is one such equation, in which both exponential and polynomial terms occur:

$$2x + x2 - 17 = 0.$$

There is unfortunately no formula that will yield the solution of this equation; note that guessing is not considered as a formula! In such instances, mathematicians fall back on graphical approaches, if a rough approximation is all that is required, or on iterative approaches, if more exact results are needed. There is nothing much to discuss as regards the graphing alternative; one simply draws the graph and does the relevant measurements. But as regards the iteration alternative, a rich scenario awaits us.

15.1 An iterative approach

We illustrate the idea of equation solving via iteration with some simple examples. The approach in each case will be to make an initial 'guess' x0 for the solution of the given equation, and then to use an appropriately chosen function f to iterate upon x0.

Example 1. Consider the linear equation $3x + 2 = 8$. We rewrite it by transposing terms as $2x = 8 - 2 - x$, i.e., as $2x = 6 - x$, and then, after division by 2, as

$$x = 6 - x\,2\,.$$

Writing f(x) for $(6 - x)/2$, the equation reads: $x = f(x)$. So the problem is to find the fixed points of f! We now construct an iteration sequence $\langle x0;f\rangle$ starting with some convenient seed, say $x0 = 0$. (This represents the initial 'guess'.) A

few terms of the orbit are displayed below.

$$x_1 = 3, \quad x_2 = 1.5, \quad x_3 = 2.25, \quad x_4 = 1.875,$$
$$x_5 = 2.0625, \quad x_6 = 1.96875, \quad x_7 = 2.0156, \quad x_8 = 1.9922.$$

The sequence appears to be converging in an up-and-down manner to a limit of 2, and this is indeed a fixed point of f, because

$$f(2) = 6 - 2\,2 = 2.$$

Note that 2 is just the solution which we seek! So the iteration has brought us very close to the exact answer. By going further down the sequence, we can improve the accuracy of our estimate to any extent that we may wish to. (Fifteen iterations yield the estimate 2.00006.)

At first encounter, this may seem to be an absurdly complicated method to solve an extremely simple equation! (Has life become so simple that we must now look for ways to complicate it?) However there are more convincing examples.

Example 2. Consider the quadratic equation x2 - 4x + 3 = 0. By moving some terms to the other side, we write it as x2 = 4x - 3, and then (after dividing by *x*) a*s*:

$$x = 4 - 3\,x.$$

Write f(x) for 4 - 3⁄x; we seek a fixed point of f. As in Example 1, we compute the orbit ⟨x0;f⟩ of any convenient seed, say x0 = 2; this is our 'guess' for the solution. (We cannot choose x0 = 0 as the expression 4 - 3⁄x is undefined when x = 0.) The resulting sequence is displayed below.

$$x_1 = 2.5, \quad x_2 = 2.8, \quad x_3 = 2.9286, \quad x_4 = 2.9756,$$
$$x_5 = 2.9918, \quad x_6 = 2.9973, \quad x_7 = 2.9991, \quad \ldots.$$

The sequence appears quite visibly to be converging to a limit of 3 (ten iterations yield the number 2.99997). This number is a fixed point of f, because

$$f(3) = 4 - 3\,3 = 4 - 1 = 3.$$

If we had chosen the initial estimate to be x0 = 4 we would have obtained:

$$x_1 = 3.25, \quad x_2 = 3.077, \quad x_3 = 3.025, \quad x_4 = 3.00826,$$
$$x_5 = 3.00275, \quad x_6 = 3.00091, \quad x_7 = 3.0003, \quad \ldots.$$

As earlier, we see a clear convergence to a limit of 3. A few more iterations will take us much closer to the limit.

It is curious that we have obtained *one* solution of the equation x2 - 4x + 3 = 0 (though tn approximate form). Is there a choice of seed that will yield the other solution, x = 1?

Example 3. Consider the equation x2 = 2. The positive solution of this equation is x = 2, so solving the equation is the same thing as numerically finding the square root of 2. To solve the equation by the iterative technique, we first rewrite it as x⁄2 = 1⁄x, then (by writing x⁄2 as x - x⁄2) as x - x⁄2 = 1⁄x, and finally as

$$x = x\,2 + 1\,x.$$

Write f(x) for x⁄2 + 1⁄x; our task is then to find a fixed point of f. The initial terms of the orbit ⟨x0;f⟩, with an initial guess of x0 = 1, are displayed here: x1 = 3⁄2 = 1.5, and

$$x2 = 17\,12 \approx 1.417,$$

$$x3 = 17\,24 + 12\,17 = 577\,408 \approx 1.4142,$$

$$x4 = 577\,816 + 408\,577 = 665857\,470832 \approx 1.41421,....$$

We see that the sequence does seem to converge to 2, and quite rapidly too.

Question. How might the method be adapted to give the square roots of numbers other than 2?

Example 4. Consider the cubic equation x3 - x - 100 = 0. We write the equation first as x3 = x + 100 and then, after taking cube roots, as

$$x = (x + 100)1⁄3$$

Denote the function (x + 100)1⁄3 by f(x); we seek a fixed point of f. As the computations involve cube roots, we use a calculator or computer. Some terms of the orbit ⟨ 1; f ⟩ are displayed below.

$$x_0 = 1.0, \quad x_1 = 4.657, \quad x_2 = 4.7126, \quad x_3 = 4.71338,$$
$$x_4 = 4.7134, \quad x_5 = 4.7134, \quad x_6 = 4.7134, \quad \ldots.$$

The sequence appears to be converging very rapidly to the limit 4.7134. Working to five decimal places (rather than the three decimal places shown above) gives the limit as 4.71339, and this is indeed "very nearly" a solution of the given equation, for

$$4.7133\ 93 \approx 104.71339 = 100 + 4.71339.$$

If we choose the initial estimate to be x0 = 100 (which looks like a terrible guess!), we obtain the sequence

100, 5.84804, 4.73036. 4.71365, 4.7134,
4.7134. 4.7134. 4.7134, 4.7134, ...

Remarkably, even after starting so far away from the desired solution, we have succeeded in reaching its vicinity in just four steps.

Example 5. We consider a fourth degree equation this time:

$$x4 - x3 - x2 - x - 1 = 0.$$

We rewrite the equation first as x4 = x3 + x2 + x + 1, and then as

$$x = (x3 + x2 + x + 1)1/4$$

Let f(x) denote the function (x3 + x2 + x + 1)1/4; we seek a fixed point of f. Some terms of the orbit ⟨1;f⟩ (we have chosen a initial guess of x0 = 1 for the solution) are displayed below:

$x_1 = 1.4142$, $x_2 = 1.6405$, $x_3 = 1.7669$, $x_4 = 1.8377$,
$x_5 = 1.8773$, $x_6 = 1.8995$, $x_7 = 1.9119$, $x_8 = 1.9188$,
$x_9 = 1.9227$, $x_{10} = 1.9248$, $x_{15} = 1.9274$, $x_{20} = 1.9276$,
$x_{25} = 1.9276$, ..., ..., ...

The sequence appears to be converging to a limit of about 1.9276. If we substitute 1.9276 for x in the expression x4 - x3 - x2 - x - 1, we obtain 0.0000481, which looks "reasonably close" to 0, so 1.9276 can be taken as a good approximation to the solution. Further computations show that a better approximation is 1.927562. (Note that there are other solutions to the

equation; to be specific, there is a solution close to - 0.7748. How could we find these solutions iteratively? What should be the choice of seed?)

The rate of convergence was rather slow in this example. However by rewriting the equation in different ways, the rate can be speeded up considerably.

Example 6. The equations considered till now have all been reducible to "nice." polynomial equations. But the iteration approach yields results even for equations where exponential and polynomial occur together. For instance, let us consider the equation

$$10x = x2 + 96,$$

whose solution may be found by simple guesswork to be x = 2. We could now ask whether we can find its solution in some systematic way. Indeed we can; but we must first rewrite the equation as

$$x = \log10(x2 + 96).$$

Let f(x) = log10(x2 + 96). Iterating f on an initial seed value of x0 = 0, we get the following results:

$$x1 = 1.9823, x2 = 1.9997,$$

$$x3 = 1.999995, x4 = 1.99999991, \ldots.$$

It is quite evident that the sequence has a limit of 2. So here too the iteration approach has yielded the desired result.

Exercises

15.1.1 Use the iteration method to solve the following two equations:

a. x5 - x - 1000 = 0;

b. x7 - x4 - x - 10000 = 0.

15.1.2 Find ways of rewriting the equation

$$x4 - x3 - x2 - x - 1 = 0$$

so that the iterative approach yields a solution faster than in Example

5, above. You may want to use the following *BASIC* program to assist you in your computations.

```
REM To solve x∧4 -x∧3 -x∧2 -x - 1 = 0
CLS
DEFDBL X-Y
nmax = 20 : x = 1
PRINT "f(x) = x∧4 -x∧3 -x∧2 -x - 1"
PRINT
PRINT "x","f(x)"
FOR i = 1 TO nmax
    y = x∧4 -x∧3 -x∧2 -x - 1
    x = (x∧3 + x∧2 + x + 1)∧(1/4)
PRINT x,y
NEXT i
```

The line x = (x∧3 + x∧2 + x + 1)∧(1/4)" is under your control; explore what happens when different expressions are substituted for the expression on the right side.

15.1.3 Estimate the solution of the equation x1/2 - x1/3 = 100 to an accuracy of four decimal places. (First write the equation in a more convenient form.)

15.1.4 [*] Estimate the solution of the equation xx+1 = (x + 1)x to an accuracy of four decimal places.

15.2 It looks much too arbitrary!

You may protest at this point, with some outrage, that the manner of writing the equation so as to obtain a "good" iteration function f(x) seems much too unclear, and indeed appears perfectly arbitrary. For instance, the equation 3x + 2 = 8 could as well have been written as

$$x = 6 - 2x,$$

in which form the iteration method fails miserably. To see why, let us examine compute the sequence $\langle 1; f \rangle$:

$$\langle 1,4, -2,10, -14,34, -62,\ldots \rangle.$$

We are clearly not getting even remotely close to the answer (which is 2 in this case). Even if we start very close to the answer, say at 2.01, things do not seem to go right; we get the sequence

$$\langle 2.01,1.98,2.04,1.92,2.16,1.68,2.64,\ldots \rangle.$$

Observe that we are steadily drawing away from the desired answer, instead of towards it.

The equation x2 - 4x + 3 = 0

The method also fails miserably for the equation x2 - 4x + 3 = 0 if we write it in the form

$$x = x2 - 3x + 3.$$

To see why, write g(x) for x2 - 3x + 3. Some terms of the orbit $\langle 4; g \rangle$ are displayed below:

$$x0 = 4,x1 = 7,x2 = 31,x3 = 840,x4 = 703083,....$$

We are caught in runaway inflation! Better results are obtained with x0 = 2:

$$x0 = 2,x1 = 4 - 6 + 3 = 1,x2 = 1,x3 = 1,....$$

As we have landed right on one of the solutions of the equation, the question of convergence does not arise. With x0 = 2.1, however, convergence is exasperatingly slow, and there is even a hint of a 2-cycle! Some terms of the orbit $\langle 2.1;g \rangle$ are displayed below.

$$\begin{array}{lll} x_1 = 1.1100, & x_2 = 0.9021, & x_3 = 1.1075. \\ x_{10} = 0.9093, & x_{11} = 1.0989, & x_{12} = 0.9109. \\ x_{50} = 0.9304, & x_{51} = 1.0744, & \ldots. \end{array}$$

Strangely. we do not seem to be “getting anywhere’ at all! Further computations only confirm this suspicion: we find that x100 ≈ 0.9433 and

x101 ≈ 1.0599. Does the orbit really converge to a 2-cycle? Actually it does not —but to see why we need to take a much closer look at the function g. Chapter 16 has some material on how this rather challenging question can be tackled. (It is again studied in Volume II.)

☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆

However, successful results are obtained once again by writing the equation first as 4x = x2 + 3, and then as

$$x = x2 + 3\ 4\ .$$

Let k(x) = (x2 + 3)⁄4. Some terms of the orbit ⟨2.0;k⟩ are displayed below.

$$\begin{array}{llll} x_1 = 1.75, & x_2 = 1.516, & x_3 = 1.324, & x_4 = 1.188, \\ x_5 = 1.103, & x_6 = 1.054, & x_7 = 1.028, & x_8 = 1.014, \\ x_9 = 1.007, & x_{10} = 1.004, & x_{11} = 1.002, & x_{12} = 1.001. \end{array}$$

The sequence appears to be converging to a limit of 1. Note that this time we have stumbled, unexpectedly, upon the *other* solution of the equation!

The equation x2 - 2x + 1 = 0

Another interesting example is provided by the equation

$$x2 - 2x + 1 = 0,$$

whose only solution is x = 1 (strictly speaking, this is a "double" root—two roots have coalesced into one). We write the equation as x = h(x), where

$$h(x) = x2 + 1\ 2\ .$$

We shall compute the orbits of 0.9 and 1.1 under iteration by h. A few sample values are given below.

With a seed value of x0 = 0.9 we find that

$$\begin{array}{lll} x_{10} = 0.93427, & x_{20} = 0.95089, & x_{30} = 0.96075, \\ x_{100} = 0.98358, & x_{200} = 0.99101, & x_{300} = 0.9938, \end{array}$$

and we observe an extremely slow (but definite) convergence to the desired solution of 1. But if the seed value is x0 = 1.1 then we get totally different results!

$$x_5 = 1.13104, \quad x_{10} = 1.18844, \quad x_{15} = 1.32614,$$
$$x_{20} = 1.95527, \quad x_{25} = 208.278, \quad x_{26} = 21690.3,$$
$$x_{27} = 2.35235 \times 10^8, \quad \ldots \quad \ldots.$$

The sequence explodes! If we choose a seed value of x0 = 1.01 we see exactly the same phenomenon, though more computation is required:

$$x_{10} = 1.01052, \quad x_{20} = 1.0111, \quad x_{30} = 1.01175,$$
$$x_{200} = 1.52051, \quad x_{202} = 1.87113, \quad x_{204} = 3.03251,$$
$$x_{206} = 13.495, \quad x_{208} = 4191.94, \quad x_{210} = 3.85987 \times 10^{13}.$$

These are indeed curious results !

The equation x2 = 2

In Example 3, if we write the equation as x = 2⁄x and choose the iteration function to be f(x) = 2⁄x, the result is rather amusing: the sequence ⟨2;f⟩ is

$$\langle 2,1,2,1,2,1,\ldots \rangle.$$

We are caught in a 2-cycle!

The equation 10x = x2 + 96

Lastly, we consider the equation of Example 6; we could also write the equation a.s

$$x = 10x - 96.$$

If we start the iteration with x0 = 2 then of course we stay at the same point for ever. If however we use the seed value x0 = 3, then we get

x1 = 103 - 96 = 904 = 30.0666,

x2 = 1030.0666 - 96 ≈ 1015,

and we have already lost the game! The same thing happens even if we use a seed value of 2.1, or 2.01, or 2.001.

Why is the outcome so unpredictable?

A question that poses itself quite naturally at this point is:

Why should the outcome of an iteration be so dependent on how the

equation is written? Why does the iterative method work so spectacularly well in some instances, and so disastrously in others.?

Much research has been done on this and related questions. Some light will be shed on the matter in Chapter 16, which examines the issue from a graphical viewpoint. (It is again studied in Volume II, using calculus.)

Chapter 16

Cobwebbing

In this chapter we describe how graphs can help in understanding the limiting behaviour of the iteration sequence

$$x, f(x), f(2)(x) = f \circ f(x), f(3)(x) = f \circ f \circ f(x), \ldots,$$

where f is a given function defined on set of the real numbers R; specifically, whether the sequence of iterates converges or not. We shall see how the condition for convergence has a natural and simple graphical interpretation. Our approach will to some extent help answer the questions posed in Chapter 15, where we had asked why, in using iterative methods for solving equations, we meet great success in some situations, and total failure in other situations; we had wondered then whether we could have anticipated what was going to happen. Strictly speaking, we need the calculus for our study, but we shall make do without it for now. (A calculus based study will be done in Volume II.)

We assume for simplicity. that the iteration function f is *smooth*; i.e., it does not have points where the graph is broken or "poky". We also assume that f has at least one fixed point. (If not, then the orbit $\langle x0;f \rangle$ will not converge for any x0.)

We now compute the orbit $\langle x0;f \rangle$ for some choice of seed x0 and trace its progress on a graph. The effect is best seen by studying specific examples, and we proceed to do just this.

Notation. Throughout our discussion we shall use the symbol α to denote a fixed point of f (so $f(\alpha) = \alpha$), the symbols $\mathcal{J}$ and $\mathcal{F}$ to refer to the graphs of $y = x, y = f(x)$ respectively, where f is the iteration function under study; the two

graphs intersect one another at the point (α,α). (If there is more than one fixed point, we use β,γ, etc., to refer to them.)

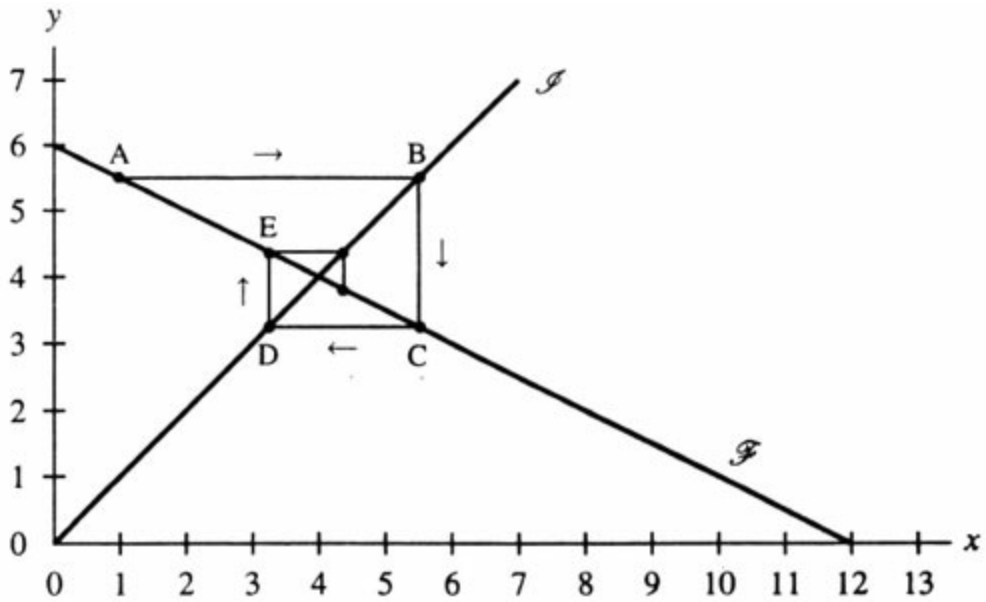


Figure 16.1. *Graphs of* y = 6 -X 2 *and* y = x

16.1 Various examples

Example 1. Let the iteration function be

$$f(x) = 6 - x^2.$$

Solving the equation x = f(x) we get the solution x = 4; so the fixed point is α = 4.

With x0 = 1, the orbit ⟨x0;f⟩ is

$$\langle 1.0, 5.5, 3.25, 4.375, \ldots \rangle.$$

The relevant diagram is displayed in Figure 16.1; $\mathcal{J}$ and $\mathcal{F}$ are the lines y = x and y = 6 -1 2x, respectively; they meet at (α,α) = (4,4).

The seed x0 = 1 corresponds to the starting point A on $\mathcal{F}$. Since x1 = 5.5, we seek the point on $\mathcal{F}$ with x-coordinate 5.5; we locate this by moving horizontally from A to B (on $\mathcal{J}$), then moving vertically ("downwards") from B to C (on $\mathcal{F}$). The first step of the iteration is thus the passage from A to C (via B).

We now repeat the process: we move horizontally from C to D, then

vertically ("upwards") from D to E, with D on $\mathcal{J}$ and E on $\mathcal{F}$ the second step of the iteration is the passage from C to E, via D. The mechanism of the procedure should be clear by now. The iteration now takes us to (4.375, 3.813), then to (3.813, 4.094), and so on. A few more steps will serve to demonstrate, visually and quite convincingly, that the sequence of moves takes us rapidly to the point (α,α) = (4,4) where $\mathcal{F}$ meets $\mathcal{J}$.

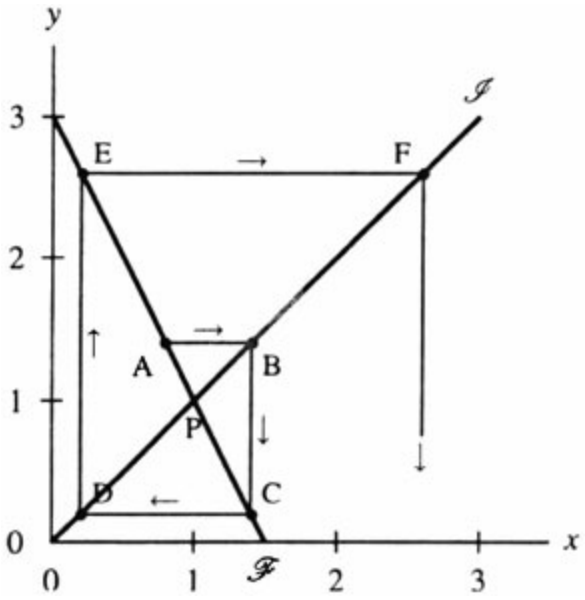


Figure 16.2. *Cobwebbing with* y = 3 - 2x

Remark. The "cobwebby" appearance of the graph should suggest vividly why the technique is referred to as **COBWEBBING.** It should be clear that cobwebbing can be done for any function f; we draw the graphs $\mathcal{F}$ and $\mathcal{J}$, select a starting point A on $\mathcal{F}$, draw a horizontal line through A to meet $\mathcal{J}$ at B, then a vertical line through B to meet $\mathcal{F}$ at C; then we repeat the procedure, starting from C. The question now is: under what circumstances will this iterated procedure take us to the point P(α,α) where $\mathcal{F}$ meets $\mathcal{J}$?

Example 2. Here is an example where convergence does *not* take place. Let

$$f(x) = 3 - 2x.$$

Solving the equation x = f(x), we get the solution x = 1; the fixed point is α = 1.

Let x0 = 0.8; then the starting point on $\mathcal{F}$. is A = (0.8,1.4). The iteration now takes us in succession to the following points on $\mathcal{F}$: C = (1.4,0.2), then to

E = (0.2,2.6),(2.6, - 2.2),(-2.2,7.4), and so on. It is visually quite clear that we are moving *away* from the desired point P(α,α) = (1,1); see Figure 16.2.

Example 3. In Chapter 15, we observed how iterative techniques can be used to numerically solve certain types of equations, e.g., the equation

$$x3 + x - 100 = 0.$$

We take a fresh look at this particular equation. We first write it as x = f(x), with f(x) = 100 - x3. The relevant graphs are shown in Figure 16.3 (the axes are not drawn to the same scale). The point P = (α,α) which is the point of intersection of $\mathcal{F}$ and $\mathcal{J}$ is close to (4.6, 4.6), because 4. 63 + 4.6 = 101.936 ≈ 100.

If we start the iteration with x0 = 4, we soon find ourselves in distant lands:

$$x1 = 100 - 43 = 36,x\ 2 = 100 - 363 = -46556,....$$

Even if we start much closer, say with x0 = 4.5, we move away very rapidly:

$$x1 = 100 - 4.53 = 8.875,x\ 2 = 100 - (8.875)3 \approx -599.045.$$

and |x3| is still larger. So here the iteration takes us away from the desired point P = (α,α) rather than towards it, even when we start very close to P. The reader may verify that this happens even if the seed value is x0 = 4.6.

However the equation can be written in other forms, e.g, as x3 = 100 - x and therefore as x = g(x), where

$$g(x) = 100 - x3.$$

The graph of g is shown in Figure 16.4 (as earlier, the two axes have different scales), along with a small fragment of the cobweb. (The portion of the graph beyond x = 10 has not been shown, as it does not feature in the cobweb.)

We start the iteration with x0 = 0 and obtain the following sequence.

$$x1 = 1003 \approx 4.6416,$$

$$x2 \approx 95.35843 \approx 4.5686,$$

$$x3 \approx 96.43143 \approx 4.5857,$$

$$x4 \approx 95.41433 \approx 4.5695\ldots.$$

The spiral zeroes very rapidly to the point of intersection, and we find the solution of the equation to be approximately 4.56978.

16.2 Dependence on slope

Visually, it is not hard to anticipate that convergence depends quite critically on the *slope* of $\mathcal{F}$ at the point P(α,α) where $\mathcal{F}$ meets $\mathcal{J}$. If the slope exceeds 1 in absolute value, then the angle between $\mathcal{F}$ and $\mathcal{J}$ is too big, and each step carries us further away from P rather than towards it. (If we ever get close to P —"by chance." so to speak—we get repelled away.) In Figure 16.5 we see a depiction of this behaviour.

On the other hand if the slope is less than 1 in absolute value, then the angle enclosed between $\mathcal{F}$ and $\mathcal{J}$ is acute, and convergence to the fixed point P does indeed take place; see Figure 16.6.

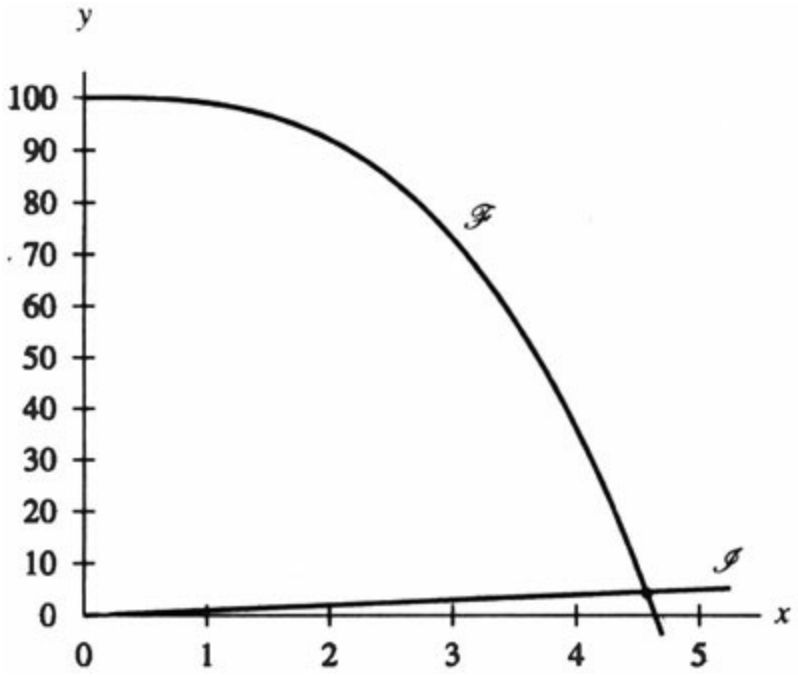


Figure 16.3. *Cobwebbing with* f(x) = 100 - x3

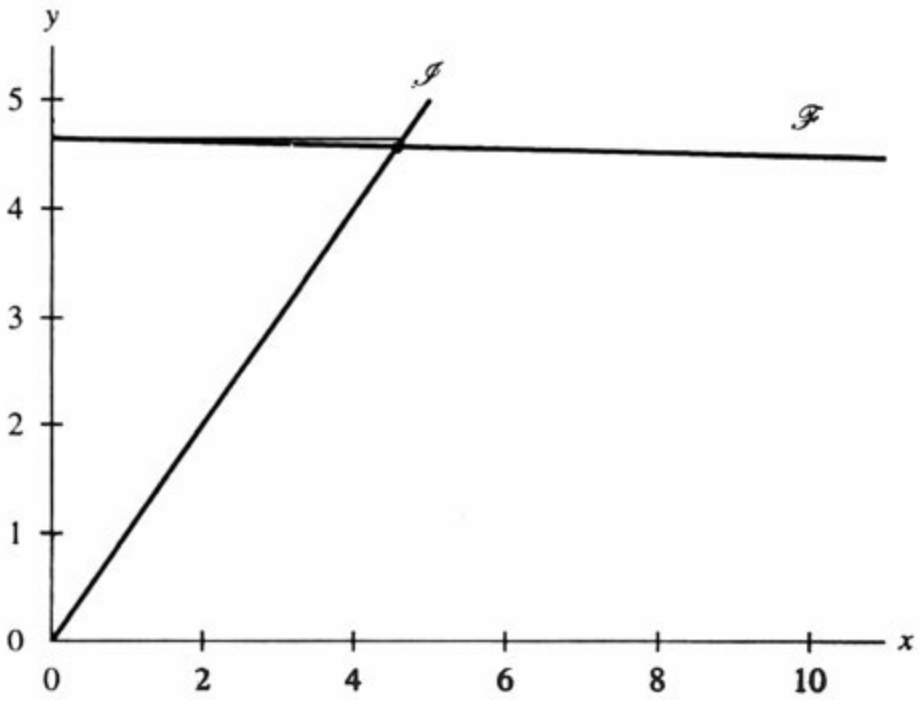


Figure 16.4. *Cobwebbing with* g(x) = (100 - x)1/3

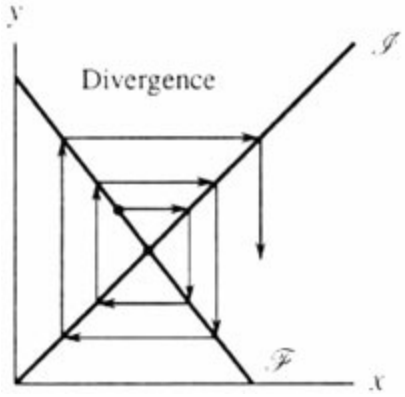


Figure 16.5. |slope| > 1

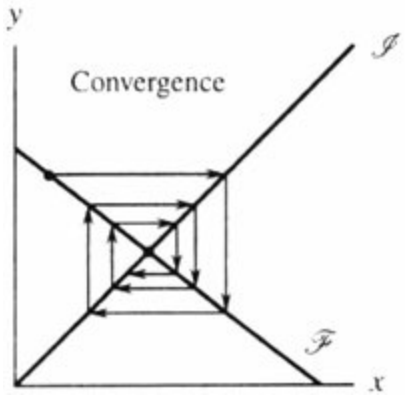


Figure 16.6. |slope| < 1

What if the slope of f **at** P **is** 0**?**

These comments lead us to anticipate that the smaller the slope of f at P, in absolute value, the faster will be the rate of convergence. We might expect therefore that a slope of zero at P(α,α) will make for very impressive results; and indeed this is the case. The next example demonstrates this behaviour most powerfully.

Example 4. We consider the function *f* given by:

$$f(x) = x\,2 + 1\,x.$$

The equation f(x) = x yields

$$x = x\,2 + 1\,x, \therefore x\,2 = 1\,x, \therefore x2 = 2,$$

so x = ±2. Let α denote the positive solution, α = 2. It may be shown, via the use of calculus, that the slope of f at α is 0. (In Figure 16.7 we have sketched the graph of f, and we are able to visually confirm that the slope of f at the point of intersection of the two graphs is 0.)

Let x0 = 1; the iterates of x0 under action by f are computed below.

x0 = 1.

x1 = 1.5,

x2 ≈ 1.41667.

x3 ≈ 1.41422.

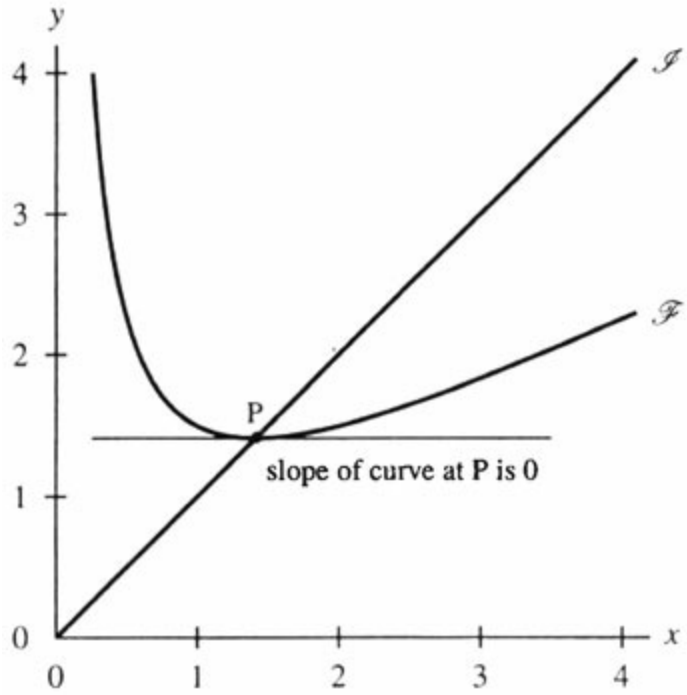


Figure 16.7. *The iteration* x↦x 2 + 1 x

x4 ≈ 1.41421356,

x5 ≈ 1.4142135623730950488.

It tums out that x5 agrees with 2 to all the digits shown above! If we carry the computations to still greater accuracy, we find the following:

x5 ≈ 1.41421356237309504880168 96,

2 ≈ 1.41421356237309504880168 87,

and |x5 -2|≈ 10-24. The rate of convergence is astonishing! The underlying reason is simply the fact that the slope of f(x) = x⁄2 + 1⁄x at x = 2 is 0.

Remark. It may be shown that the rate of convergence here is "quadratic"; that is, the error at each stage is roughly equal to the *square* of the error at the preceding stage. This accounts for the extremely rapid convergence observed above.

16.3 The case when slope equals ± 1

We still need to find out what happens when the slope is equal to 1 in absolute value.

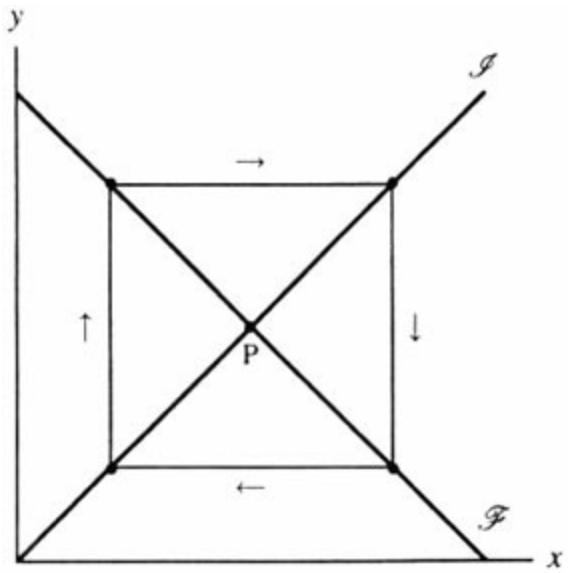


Figure 16.8. *Trapped in a 2-cycle*

If the slope of f at P is - 1, then the angle between $\mathcal{F}$ and $\mathcal{J}$ is 90◦, so we can expect at least an approximation to a 2-cycle. This is so because $\mathcal{F}$ and $\mathcal{J}$ intersect roughly like the diagonals of a square. The cobweb now resembles the sides of the square, and we simply move around the boundary of the square, without any significant progress towards the center, P; so we could get trapped in a 2-cycle. (See Figure 16.8.) This happens, for instance, if f(x) = 1 - x and x0 is any seed other than the fixed point 1/2; e.g., x0 = 1⁄3 yields the 2-cycle ⟨1⁄3,2⁄3⟩. But this will not always happen.

16.4 The iteration x↦1 - 3x2⁄4

A much more interesting example is provided by the function

f(x) = 1 -3x2 4 .

We proceed to study this iteration in some detail. The equation f(x) = x yields

$$x = 1 - 3x^2 4 \text{ ,} \therefore 3x2 + 4x - 4 = 0, \therefore (3x - 2)(x + 2) = 0.$$

The solutions are x = 2⁄3 and x = -2. Our interest lies in the positive solution, and we write α = 2⁄3.

The use of calculus shows that the slope of f at 2/3 is - 1. Let us see if the expected behaviour takes place. Here is the outcome when x0 = 0.9 (we give only a few selected values):

x10 = 0.845,x11 = 0.465,

x30 = 0.8,x31 = 0.52,

x100 = 0.751,x101 = 0.577.

There is a definite suggestion of a 2-cycle, but the appearances are deceptive; in fact there is slow convergence to 2/3 (only, it is very slow). The reason however is not immediately obvious. Here is one way of seeing why—by examining the second iterate f2. We find that

$$f2(x) = 1\ -3\ 4\ 1\ -3x2\ 4\ 2 = -27x4 - 72x2 - 16\ 64\ ,$$

and the equation x = f2(x) simplifies, after much simplification, to

$$27x4 - 72x2 + 64x - 16 = 0.$$

Factorization now yields the equation

$$(x + 2)(3x - 2)3 = 0,$$

whose solutions are x = -2 and x = 2⁄3. These are however just the fixed points of f! So f2 has no fixed points other than those of f, and therefore f has no 2-cycle.

The graph reveals what is happening. Denoting the graph of f2 by $\mathcal{F}2$, we find (see Figure 16.9) that $\mathcal{F}2$ is tangent to $\mathcal{J}$ at x = 2⁄3; and that—this is crucially important—it lies *above* $\mathcal{J}$ on the left side and *below* it on the right side. Visually, it is not hard to see from the graph that if we start the cobweb at a point to the right of P, the point of intersection, we shall slowly drift towards P. Exactly the same thing happens when we start at a point to the left of P. It follows from this that 2/3 is "weakly attracting" from both sides.

Proof of convergence

The fact that the orbit ⟨x0;f⟩ converges to 2/3 may be shown rigorously as follows. (The logic is fairly intricate, so brace yourself before plunging ahead!)

We first perform a shift of origin: we work with the sequence ⟨un⟩ defined by un = xn - 2⁄3 rather than with ⟨xn⟩; this means that we have to show that the sequence ⟨un⟩ converges to 0. The equation connecting un+1 with un is given by

un+1 + 2 3 = 1 -3 4 un + 2 32, ∴ u n+1 = -un -3un2 4 .

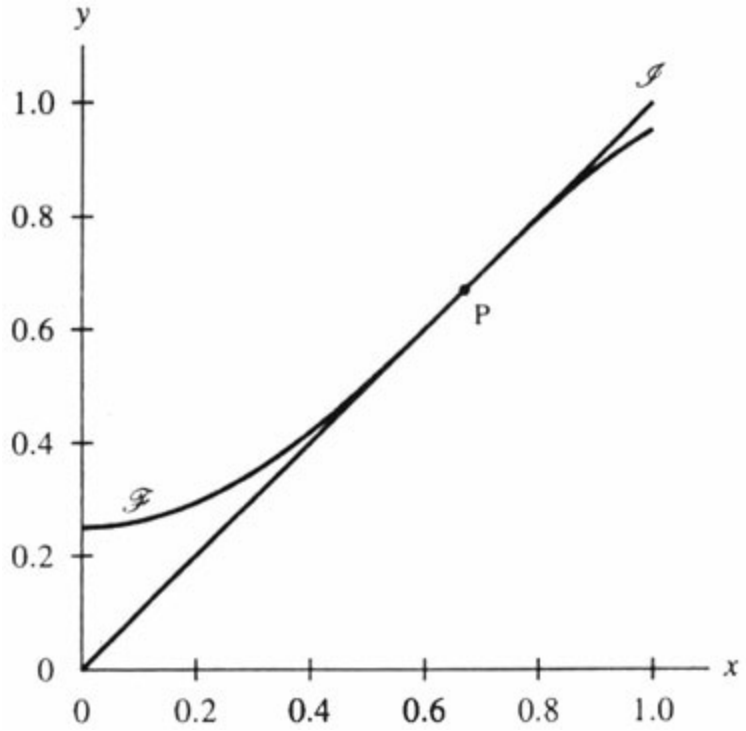


Figure 16.9. *Graph of* f2(x) = -27x4 - 72x2 - 16 64

So the iteration on u is governed by the function g, where

g(u) = -u -3u2 4 .

The fixed points of g are 0 and - 8⁄3 (corresponding to the fixed points 0 and – 2 of f); our interest of course lies in the first value.

The two-step iteration on u, i.e., un↦un+2, or u↦g(g(u)), is governed by the function g2 = g ◦ g given by

g2(u) = --u -3u2 4 -3 4 -u -3u2 4 2 = u -9u3 8 -27u4 64 .

We now establish two basic results.

- *If* 0 < u < 1 2 *then* 0 < g2(u) < u.

  The relation g2(u) < u is clear, as the quantities 9u3⁄8 and 27 u4⁄64 exceed 0 (since u > 0). To show why g2(u) > 0, we rewrite the relation as

  u(64 - 72u2 - 27u3) 64 > 0.

  The numerator of the above fraction factorizes in a convenient manner,

and

we rewrite the inequality as

$$u(3u + 4)(9u2 + 12u - 16) < 0.$$

If $0 < u < 1/2$, then the factors u and 3u + 4 exceed 0. The roots of the quadratic factor 9u2 + 12u - 16 (i.e., the points where it takes a value of 0) are

$$- 12 \pm 122 + 576\ 18\ , \text{ i.e., } - 2.1574 \text{ and } 0.824.$$

So the quantity 9u2 + 12u - 16 is negative when $-2.1574 < u < 0.824$. As the interval from 0 to 1/2 lies within this interval, it follows that 9u2 + 12u - 16 is negative for $0 < u < 1/2$. This yields the stated result.

- If $-1\ 2 < u < 0$, *then* $u < g2(u) < 0$.

  This holds because u(3u + 4) and 9u2 + 12u - 16 are both negative when $- 1/2 < u < 0$.

It follows from these two claims that if u0 lies between 0 and 1/2, then the sequence

$$u0,u2,u4,u6,u8,...$$

is steadily decreasing (*monotonic decreasing*, to use the technical term); also, the sequence can never go below 0. (it is "bounded below by 0," to use the technical term). It is known from higher analysis (we shall say a bit more about such matters in Volume II) that such a sequence must have a limit, and here the limit can only be 0. Likewise, if we have $- 1/2 < u0 < 0$ then the sequence

$$u0,u2,u4,u6,u8,...$$

is steadily increasing (*monotonic increasing*), and it is bounded above by 0; there- fore it has a limit, and the limit must be 0. So if $|u0| < 1/2$ then the iterates of u0 under iteration by g converge to 0, implying that the iterates of x0 under iteration by f converge to 2/3.

16.5 Two more case studies

In Chapter 15 we had closely analyzed the two iterations

$$x \mapsto g(x) = x2 - 3x + 3, x \mapsto h(x) = x2 + 1\ 2\ .$$

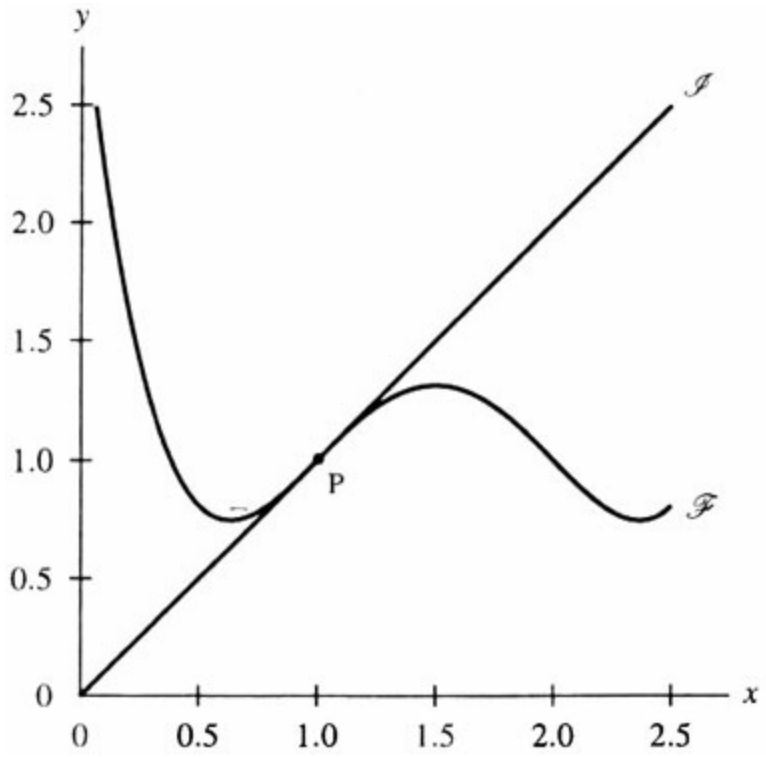


Figure 16.10. *Graph of* g2(x) = x4 - 6x3 + 12x2 - 9x + 3

We now take a closer look at these two iterations.

Observe that α = 1 is a fixed point for both g and h (for g(1) = 1,h(1) = 1). The use of calculus shows that the slope of g at α is - 1, whereas the slope of h at this point is 1. We observed earlier that the orbit ⟨2.1;g⟩ "hovers contentedly" about the fixed point 1, with little apparent movement either towards or away from this point. For the function h we observed that the orbit ⟨0.9;h⟩ shows a very slow convergence to 1, whereas the orbit ⟨1.1;h⟩ shows a very slow divergence away from 1. Let us see now whether these observations could have been anticipated from the respective graphs.

We sketch the graphs of the second iterates (g2 and h2); see Figure 16.10 and Figure 16.11. We find in each case that the graph of the second iterate is tangent to the line y = x at the point (α,α). But there is a crucial difference between the two graphs!

For the function g(x) = x2 - 3x + 3, we find that the graph of g2 lies above $\mathcal{J}$

to the left of P, and below $\mathcal{J}$ to the right of P (Figure 16.10). So the slow convergence of the cobweb towards P is a natural and inevitable outcome.

On the other hand, for the function h(x) = (x2 + 1)⁄2, we find that the graph of h2 lies above $\mathcal{J}$ on *both* sides of P (Figure 16.11). So it is natural to expect that if we start to the left of P, there will be a slow convergence of the cobweb towards P, whereas if we start to the right of P, there will be a slow divergence away from P.

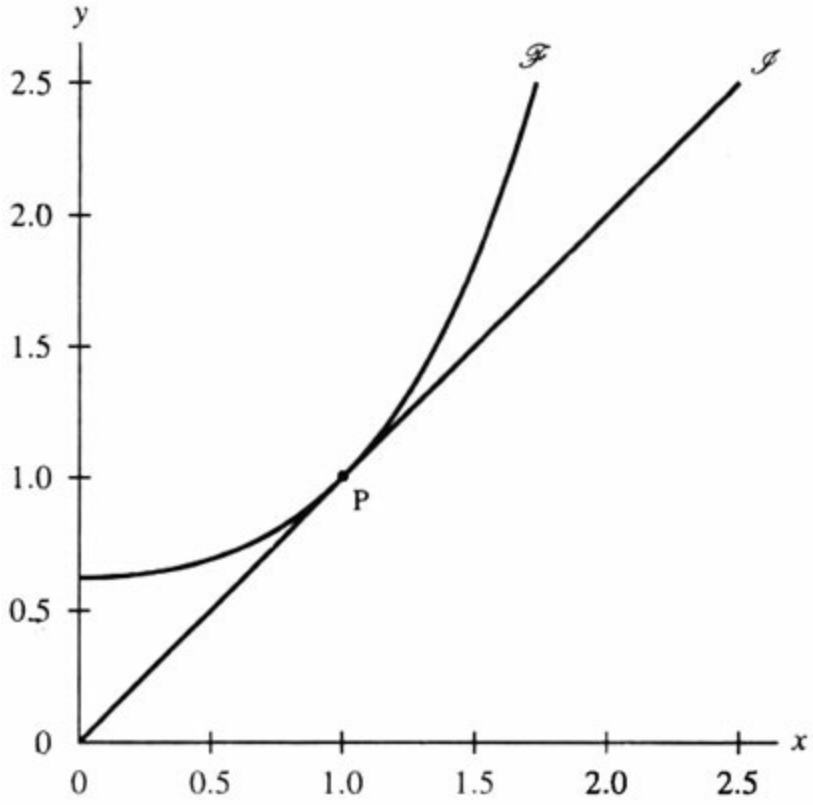


Figure 16.11. *Graph of* h2(x) = x4 + 2x2 + 5 8

The reader should devise more examples of this sort, where the slope of the iteration function f at the point where y = x (i.e., where it intersects $\mathcal{J}$) is - 1.

16.6 Concluding remarks

It is indeed quite remarkable that we are able to anticipate so accurately just how these iterations behave, simply by studying the relevant graphs.

Given our comparative success in this area, there is one further item that we may feel justified in asking for—an estimate for the *speed* of convergence,

in the cases where convergence take place. It turns that this too can be done. But this requires much more analytical machinery (some calculus, and a deeper understanding of limits). We shall take up the matter in Volume II.

Chapter 17

Memorable Problems from the IMOs

In this chapter we present some memorable problems which have appeared in the International Mathematical Olympiad.

17.1 Problem IMO 1986/2

Here is a pretty problem in geometry from IMO 1986 (held in Warsaw, Poland):

> *A triangle* A1A2A3 *and a point* P0 *are given in a plane. We define* As = As-3 *for integers* s ≥ 4, *and construct a sequence of points*
>
> $$P0,P1,P2,P3,P4,\ldots$$
>
> *such that* Pk+1 *is the image of* Pk *under a clockwise rotation centered at the vertex* Ak, *through an angle of* 120◦(k = 0,1,2,…). *Prove that if* P1986 = P0, *then the triangle* A1A2A3 *is equilateral.*

Solution

As it happens, the only reason why the integer 1986 is used in the problem is that 1986 is divisible by 3! (Those who are familiar with the IMO will know that problem posers often use the year of the examination in this manner.)

We need some results from the *geometry of transformations*; specifically, about the way that rotations and translations compose with one another. The main facts needed are given below.

- Let A and B be two given points and α and β two given angles. Let f and g be the rotations centered at A and B respectively, through angles of α

and β (the angles are signed; angles measured in the counter-clockwise direction are considered to be positive, and angles measured in the clockwise direction are considered to be negative). Then for the composite map g ◦ f (which means, "*first do* f, *then do* g"), the following is true: *If* α + β *is a multiple of* 360o, *i.e., a multiple of a complete rotation, then* g ◦ f *is a translation*; *otherwise*, g ◦ f *is a rotation centered at the unique fixed point of the map* g ◦ f.

This result may be extended inductively to the case of several such rotation mappings: if the algebraic sum of the angles of rotation is a multiple of 360◦, then the composite map is a translation; and in all other instances it is a rotation.

- If f,g are translations, then g ◦ f is a translation.

These results are easily proved, but we shall not go into the proofs here. (The interested reader is referred to I M Yaglom's highly readable three volume set on this topic, *Geometric Transformations* I,II,III, published by the Mathematical Association of America.)

In the present problem, let f,g and *h* refer respectively to the rotations centered at A1,A2 and A3, each through an angle of 120◦ (i.e., 1/3 of a full turn), measured clockwise. Let T be the composite operation *h* ◦ g ◦ f; then T is a translation, because the sum of the angles of rotation is 360° and corresponds to a full turn. Clearly

$$P1986 = T(662)(P\ 0),$$

where T(662) refers to the operation T repeated 662 times (note that 662 = 1986⁄3) . Since T is a translation, so is T(662). The equality P1986 = P0 implies that P0 is a fixed point of this translation. But a translation cannot have a fixed point unless it is just the identity map (i.e., unless it is a "trivial" translation). It follows that T(662) is the identity map, and therefore so is T itself.

This means that *h* is the inverse of the composite map g ◦ f. But g ◦ f is a rotation, through 240o, about the unique fixed point of g ◦ f. It is readily shown that the fixed point of gof is the vertex of the equilateral triangle constructed on A1A2 a*s* base and oriented suitably. (Please prove this for

yourself.) The map *h* must be centered at this very vertex, because *h* = (g ◦ f) (-1). It follows that △A1A2A3 is equilateral.

17.2 Problem IMO 1987/4

This problem was posed in the International Mathematical Olympiad (IMO) of 1987, held thaı year in Havana, Cuba. (It was also given as Problem 10.5.5 in Chapter 10.)

> *Show that there does not exist any function* f *from the set of natural numbers*, N, *into itself, such that* f(f(n)) = n + 1987 *for all* n ∈N.

Solution

Let k be any odd positive integer. We shall show that there is no function f from N into itself such that f(f(n)) = n + k for all n ∈N. This will settle the problem at hand, as 1987 is odd.

Let g(n) denote the function n + k. We shall suppose, with the intention of setting up a "proof by contradiction," that there does exist a function f from N into itself such that f◦ = g. We now argue as follows.

- The map f must be injective, that is, f cannot map two distinct numbers to the same number; for if f(m) = f(n), then we get

  $$f(f(m)) = f(f(n)),\ \therefore m + k = n + k,\ \therefore m = n.$$

- Consider the orbits of g. These are k in number, and they may be depicted diagrammatically thus:

  $$\begin{array}{ccccccccccc}
  1 & \longmapsto & k+1 & \longmapsto & 2k+1 & \longmapsto & 3k+1 & \longmapsto & 4k+1 & \longmapsto & \cdots \\
  2 & \longmapsto & k+2 & \longmapsto & 2k+2 & \longmapsto & 3k+2 & \longmapsto & 4k+2 & \longmapsto & \cdots \\
  \cdots & \longmapsto & \cdots & \longmapsto & \cdots & \longmapsto & \cdots & \longmapsto & \cdots & \longmapsto & \cdots \\
  k & \longmapsto & 2k & \longmapsto & 3k & \longmapsto & 4k & \longmapsto & 5k & \longmapsto & \cdots
  \end{array}$$

  Each of these k orbits extends to infinity on the right side. Note that there is no path leading into the numbers 1, 2, ..., k. Accordingly, we shall refer to these orbits as *singly-infinite*. Clearly these k singly-infinite orbits exhaust all the elements of **N**.

- Consider the orbits of f. They could be of three types:

  – *finite*, that is, the orbit is a cycle; the cycle cannot have any edges or trails "sticking out" of it because f is injective;

  – *singly-infinite*, starting from a seed value with no path leading into it;

  – *doubly-infinite*, extending indefinitely in both directions.

  Consider now the orbits of f ◦ f. Under two-fold iteration, each singly-infinite and doubly-infinite orbit of f splits into two orbits of the same type, consisting of alternate elements of the original orbit. Each **EVEN CYCLE** (i.e., a cycle containing an even number of elements) splits into two cycles, and each **ODD CYCLE** (i.e., a cycle containing an odd number of elements) remains a single cycle. (If the cycle is odd, then by taking every second element we end up enumerating all the elements, only in some different order.)

  It follows that for f ◦ f, the following numbers, *if finite*, are all even: the number of singly-infinite orbits; the number of doubly-infinite orbits; and the number of even cycles having a given cardinality.

  Therefore, if the functional equation f ◦ f = g is to be solvable for f, then g must necessarily satisfy this condition: *Each of the following numbers —if finite—must be even*: *the number of orbits that are singly-infinite, doubly-infinite, or finite and of a fixed even length*.

- The function g being considered in the present problem has k orbits of the singly-infinite variety, and no orbits of either of the other two varieties. It follows that if k is odd, such a function cannot exist.

This is a remarkable proof—it is based on considerations of parity and nothing else!

Generalization

The reader may wish to try proving the following generalization due to N Martin (MATHEMATICS MAGAZINE, December 1989, page 345, Problem 1308):

> *Let* g *be a bijection of* **N**, *and let* k *be any odd positive integer. Then there does not exist any function* f *from* **N** *into itself, such that*

$$f(f(n)) = g(n) + k \text{ for all } n \in \mathbb{N}.$$

17.3 Problem IMO 1988/6

In the IMO of 1988 (held in Canberra, Australia), the following problem appeared as # 6; it had been proposed by the Federal Republic of Germany.

Let a,b *be positive integers such that the quantity*

$$c = a2 + b2\ ab + 1$$

is an integer; *show that* c *is a square.*

A numerical instance of the hypothesis of the problem is given by (a,b) = (2,8), with c = 4; note that 4 is a square.

The paucity of correct responses received subsequently earned for the problem the title "The Most Difficult Problem Ever Posed At An IMO". (But an update is needed on this; readers who are familiar with the olympiads will know that Problem 6 of IMO 1993 (Istanbul, Turkey) and Problem 5 of IMO 1996 (Mumbai, India) are now strong contenders for the title!) Here we offer a solution to a generalized version of the problem, and then we consider a few related problems.

We shall prove the following stronger result which implies the above one.

Theorem 17.1 *Let* a,b *and* c *be positive integers such that*

$$1 \le a2 + b2 - abc \le c + 1.$$

Then the quantity a2 + b2 - abc *is a square.*

Remark. The case a2 + b2 - abc = c is equivalent to problem IMO 1988/6.

Proof of Theorem 17.1

Let d denote the value of a2 + b2 - abc; then $1 \le d \le c + 1$. We need to show that d is a square. We anchor the proof on the possible values taken by *c*.

- If c = 1, then d = 1 or 2.
  If d = 1, there is nothing to prove (for 1 is a square). The case d = 2 cannot happen, as may be shown by a simple parity-based argument: a2

+ b2 - ab is odd if at least one of a,b is odd, and is a multiple of 4 if a,b are both even.

- If c = 2, then the result follows trivially, for now d = a2 + b2 - 2ab = (a - b)2.
- From this stage onwards we shall assume that c > 2.

If d = 1, there is nothing to prove. If d > 1, we consider the curve Γd ⊂R2 defined by

$$\Gamma d := \{(x,y) : x2 + y2 - cxy = d\}.$$

This is the equation of a hyperbola symmetric in the lines y = ±x. To see why it is a hyperbola, note that the discriminant is c2 - 4, which is positive since c > 2, and the quadratic expression x2 + y2 - cxy - d cannot be factorized into linear factors, so the locus is certainly not the union of two straight lines.

Our interest lies in the lattice points on Γd, of which (a,b) is one such. (A *lattice point* is one whose *x*- and *y*-coordinates are both integers.) The crucial observation that we make is the following:

> *From any one lattice point of* Γd, *we can iteratively generate infinitely many such points.*

The "iterative generation" is accomplished as described below.

Let A(u,v) be any lattice point on the upper branch of Γd; then v > u. We move vertically down from A to the line y = x (see Figure 17.1), meeting it at B(u,u); then we move horizontally from B to Γd, meeting it at C. To find the coordinates of C, note that the image of A under reflection in the line y = x is the point A′(v,u) which lies on the lower branch of the curve. So one point of intersection of the curve and the line y = u is (v,u); that is, one root of the quadratic equation

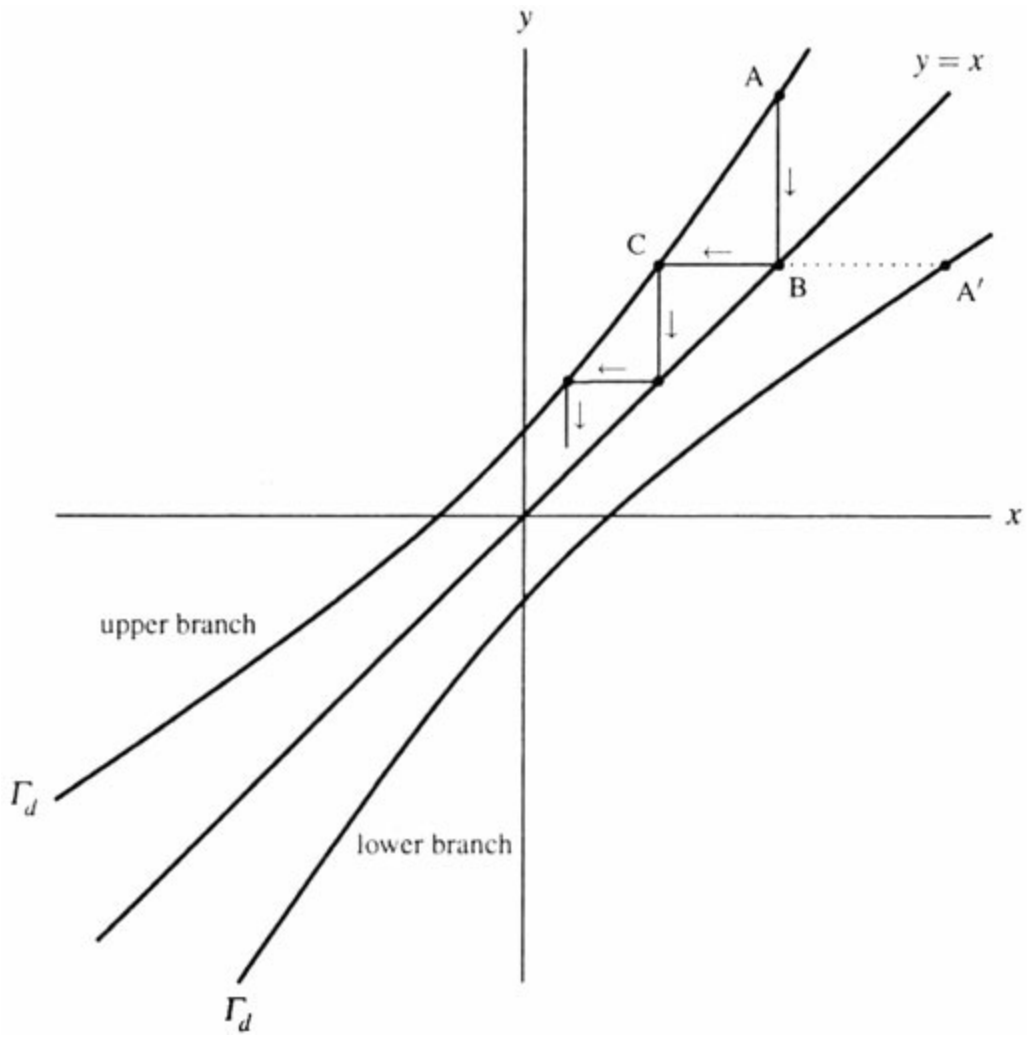


Figure 17.1. *Iterative generation of lattice points on the hyperbola* x2 + y2 - cxy = d

$$x2 - cux + (u2 - d) = 0$$

is x = v. Since the sum of the roots of the equation is c u, the other root must be x = cu - v, implying that C = (c u - v,u). Since c ,u,v are integers, it follows that C is a lattice point. Note that C and A lie on the same branch of the curve.

So we have moved from (u,v) to (c u - v,u); this is the step which we iterate. Each such step carries us from one lattice point of Γd to another one, and on the same branch. As the curve has positive slope at every point, the movement results in a strict decrease in both coordinates. The iteration results in an eventual passage into the third quadrant.

An example will make the procedure more transparent. Consider the solution x = 16,y = 42,c = 3 to the relation

$$1 \le x2 + y2 - cxy \le c + 1,x,y,c \in N.$$

The value of d = x2 + y2 - cxy is 4 in this case, so the equation of the curve Γd is x2 + y2 - 3xy = 4. Starting from (16, 42), the path taken by the iteration is the following:

$$(16,42)\mapsto(6,16)\mapsto(2,6)\mapsto(0,2)\mapsto(-2,0)\mapsto(-6,-2)\mapsto\cdots.$$

Observe the steady drift into the third quadrant.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★

Since d ≤ c + 1, the curve cannot have any lattice points in the interiors of the second and fourth quadrants; for if (x,y) were such a lattice point, then we would have

$$xy \le -1,x2 + y2 \ge 2,$$

implying that 2 ≤ x2 + y2 = cxy + d ≤ 1, which is clearly absurd.

Now it follows from the nature of the iteration that we cannot jump from the interior of the first quadrant to the interior of the third quadrant in a single step; for if v > u > 0 then the signs of the second coordinates in (u,v) and (*cu* − *v, u*) are both positive. Also, we are barred from entering the interior of the second quadrant. Therefore in order to reach the interior of the third quadrant, we must step upon both the x-axis and the y-axis, i.e., on the points

$$0,d, -d,0,$$

which consequently must be lattice points. It follows that d is a square. ❑

Remark. The features of the defining equation x2 + y2 - cxy = d that make this proof "work" are the following:

- the equation is quadratic (i.e., with degree 2);
- the equation has integral coefficients and is monic; and
- the equation is symmetric in x and y.

One often sees these properties made use of in number-theoretic contexts. (Cubic curves have their own special features, and these can be exploited too.)

A corollary

The following assertion is an easy corollary to Theorem 17.1:

Corollary 17.1 *If* a,b *are a pair of positive integers such that the quantity*

$$c = a2 + b2 + 1\ ab + 1$$

is an integer, then c *is of the form* k2 + 1 *with* k ∈N.

Examples of pairs of positive integers satisfying the stated condition are (r,r + 1), for any r ∈N, with c = 2;(2,10), with c = 5;(30,3), with c = 10; and (68, 4), with c = 17. Observe that the numbers 2, 5, 10 and 17 are all of the form k2 + 1.

More generally, one can assert that, for any given integer q, if the quantity

$$c = a2 + b2 + q\ ab + 1$$

is an integer, then c is of the form k2 + q for some integer k.

More such corollaries can be formulated, and we invite the reader to do so.

17.4 Descendants of IMO 1988/6

Problem 1988/6 has spawned numerous offspring ! We consider some of them here.

Theorem 17.2 *If* a,b ∈N *are such that the quantity*

$$c = a2 + b2\ ab - 1$$

is an integer, then c = 5.

Examples of pairs (a,b) for which (a2 + b2)/(ab - 1) is an integer are (1, 2), (1, 3), (2, 9) and (3, 14); note that c = 5 in each case.

Proof of Theorem 17.2

Let Γc denote the curve

$$\{(x,y) : x2 + y2 - cxy + c = 0\}.$$

Observe that if Γc is to be non-empty, then we must have $c \geq 3$; for if $c = 1$ or 2, then $x2 + y2 - cxy > -c$ for all $x,y > 0$. Thus we may assume that $c \geq 3$. Under this condition, Γc is a hyperbola with asymptotes

$$y = (c + c2 - 4)x\ 2\ ,y = (c\ -c2 - 4)x\ 2\ .$$

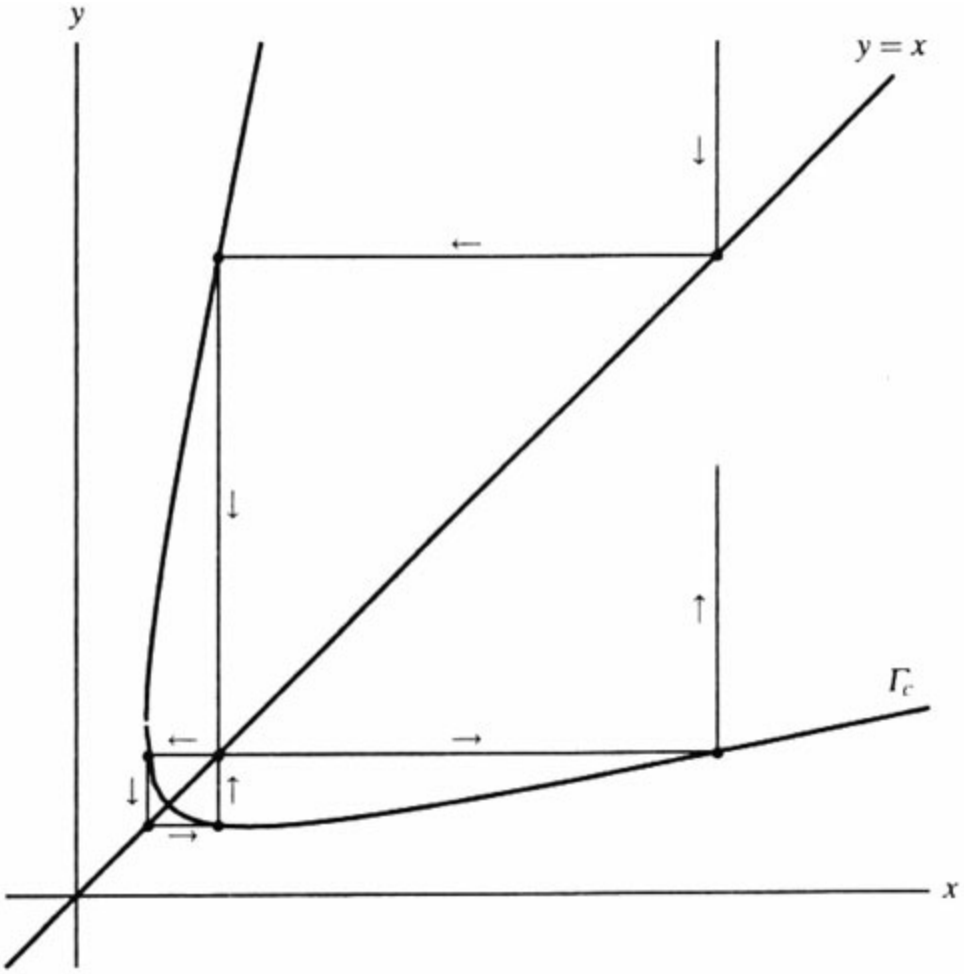


Figure 17.2. *Iteratively generating lattice points on the hyperbola* x2 + y2 - cxy + c = 0

The curve is symmetric with respect to the lines $y = \pm x$; it has two branches, one each in the first and third quadrants. One branch is shown in Figure 17.2 (this is the branch we shall be concerned with.)

The lattice points on Γc correspond exactly to the pairs of positive integers (a,b) for which $a2 + b2 = c(ab - 1)$. It is easy to verify that

$$(u,v) \in \Gamma c \Leftrightarrow (cu - v,u) \in \Gamma c.$$

This is just the reflection iteration as applied to this problem; as in Theorem 17.1, the basic step consists of a movement from (u,v) to $(cu - v,u)$ via the intermediate point (u,u) (this acts as the "stepping stone"). By iterating the procedure, we obtain from any one lattice point on Γc infinitely many such points.

For example, consider the point (9, 43) on Γ5; its orbit is given below:

$$(9,43)\mapsto(2,9)\mapsto(1,2)\mapsto(3,1)\mapsto(14,3)\mapsto(67,14)\mapsto\cdots.$$

Observe that the orbit stays on the first-quadrant branch of the curve during the course of the iteration. This will clearly always be the case (jumping from one branch to the other is not possible), and it implies that at some point the orbit "turns" around the first-quadrant vertex of Γc, namely the point

$$c\ c - 2,\ c\ c - 2.$$

If the curve is imagined to be the orbit of a comet, then this point would correspond to the perihelion of the orbit!

Since $c \geq 3$, the vertex lies in the interior of the segment connecting the lattice points (1, 1) and (2, 2). A close look at the geometry of the iteration will show that in order to turn "around the corner," we must at some stage use (1, 1) as one of the stepping stones, for it happens to be the only lattice point available for the task. *This implies that the line* $x = 1$ *intersects* Γc *at lattice points*. Therefore the roots of the quadratic equation

$$y2 - cy + (c + 1) = 0$$

are both integral. Since the discriminant of this quadratic equation is c2 - 4c - 4, it follows that c2 - 4c - 4 is a square. Now c2 - 4c + 4 = (c - 2)2 is a square too, so we have at hand a pair of squares differing by 8.

It may be shown that the only pair of squares that differ by 8 are 1 and 9; for, if we have u2 - v2 = 8, then u - v and u + v are complementary factors of 8, so (u + v,u - v) = ±(8,1),±(1,8),±(4,2) or ± (2,4). The first two possibilities do not work out, as they yield values of u and v that are not integral; the other possibilities yield the stated assertion. We conclude that (c - 2)2 = 9, ∴c - 2 = ±3. Since c > 0, we get c = 5. ❑

Theorem 17.3 *If* a,b ∈N *are such that the quantity*

$$c = a2 + b2 + 1\ ab$$

is an integer, then c = 3.

Examples of pairs (a,b) for which (a2 + b2 + 1)⁄ab is an integer are (1,1),(2,1), (5,2) and (13,5); note that c = 3 in each case.

Theorem 17.4 *If* a,b ∈N *are such that the quantity*

$$c = a2 + b2 + 6\ ab$$

is an integer, then c = 8.

Examples of pairs (a,b) for which (a2 + b2 + 6)⁄ab is an integer are (1, 1), (7, 1), (55, 7) and (438, 55); note that c = 8 in each case.

Theorem 17.5 *If* a,b ∈N *are such that the quantity*

$$c = a2 + b2 + 10\ ab$$

is an integer, then c = 12.

Examples of pairs (a,b) for which (a2 + b2 + 10)⁄ab is an integer are (1, 1), (11, 1) and (131, 11); note that c = 12 in each case.

Theorem 17.6 *If* a,b ∈N *are such that the quantity*

$$c = a2 + b2 + 8\ ab$$

is an integer, then c = 6,10 *or* 14.

Examples of pairs (a,b) for which (a2 + b2 + 8)⁄ab is an integer are (1, 1), (2, 2), (3, 1), (6, 2), (9, 1), (17, 3) and (22, 6); note that c = 6,10 or 14 in each case.

★★★★★★★★★★★★★★★★★★★★

We leave the proofs of Theorems 17.3, 17.4, 17.5 and 17.6 to the reader; the same strategy may be followed as earlier, in each case.

Theorem 17. 7 *If a and b are positive integers such that* a2 + b2 - a *is divisible by* 2ab, *then* a *is a perfect square.*

Examples of pairs (a,b) for which a2 + b2 - a is divisible by 2ab are (1,2k) for any k ∈N,(4,6),(9,6),(9,12),(16,12) and (16, 20); note that a is a perfect square

in each case.

This result once appeared as a problem in the British Mathematical Olympiad (BMO).

Proof of Theorem 17.7

Though the strategy we use in this proof is essentially the same as earlier, the exposition will be a bit different; we shall not refer explicitly to the associated graph.

Let a and b be positive integers such that a2 + b2 - a is divisible by 2ab; say

$$a2 + b2 - a = 2kab \quad (k \text{ is some integer}).$$

Note that we cannot have k = 0, because a2 + b2 - a = a(a - 1) + b2, which is positive. If k = 1, then we get a = a2 + b2 - 2ab = (a - b)2, which is a square, as required.

So we may assume in our analysis that k > 1. In the exposition below, k refers to any fixed positive integer exceeding 1 for which there exist positive integers a and b such that 2ab is a divisor of a2 + b2 - a.

We cannot have a = b; for this equality leads to - a = (2k - 2)a2, which is absurd as the quantity on the left side is negative whereas the quantity on the right side is positive. Therefore, a≠b.

Next, observe that

$$a \mid a2 - a - 2kab = b2, \therefore a \mid b2,$$

$$b \mid b2 - 2kab = a - a2, \therefore b \mid a2 - a.$$

So both b2⁄a and (a2 - a)⁄b are non-negative integers. This implies in particular that if b = 1 then a = 1 too; but this does not fit, so we must have b > 1.

Now the relation x2 + y2 - 2kxy - x = 0 may be regarded as a quadratic equation in either x or y by writing it as

$$x2 - (2ky + 1)x + y2 = 0, \quad \text{or} \quad y2 - (2kx)y + (x2 - x) = 0.$$

If we regard the first equation as a quadratic equation in x, then the product

of its roots is y2. Similarly, if we regard the second equation as a quadratic equation in y, the product of its roots is x2 - x. It follows that if the integer pair (x,y) satisfies the relation x2 + y2 - 2kxy - x = 0, then so do the pairs

$$y2\ x\ ,y,\ x,\ x2\ -\ x\ y\ .$$

The divisibility relations derived earlier show that these are in fact *pairs of integers*. Of significance too is the fact that if $0 < x < y$ then

$$y2\ x > x,0 \leq x2\ -\ x\ y < y,$$

and if $0 < y < r$ then

$$0 < y2\ x < x,x2\ -\ x\ y \geq x.$$

Let H and V denote maps ('H' for 'horizontal', ' V ' for venical'; think of the associated graph) which operate on the non-negative solutions (x,y) of the relation x2 + y2 - 2kxy - x = 0, as follows: if $y > 0$ (in which case we must also have $x > 0$), then

$(x,y)\rightarrow H\ y2\ x\ ,y,(x,y)\rightarrow V\ x,\ x2\ -\ x\ y\ .$

If $y = 0$ then we must have $x = 1$ or 0, so we complete the definition by writing:

$$(1,\ 0\ )\ \rightarrow H(0,0),(1,\ 0\ )\ \rightarrow V(1,2k),$$

$$(0,0)\rightarrow H(1,0),(0,0)\rightarrow V(0,0).$$

Note that these maps are self-inverse: $H \circ H = V \circ V =$ the identity map. (Such maps are known as *involutions*.)

The map $V \circ H$ is given by $(x,y) \rightarrow V\ oH\ (x',y')$ where

$$x' = y2\ x\ ,y' = y4⁄x2\ -\ y2⁄x\ y = y3\ x2\ -y\ x,$$

and the map $H \circ V$ is given by $(x,y) \rightarrow HoV\ (x'',y'')$ where

$x'' = (x2\ -\ x)2⁄y2\ x = x(x\ -\ 1)2\ y2\ ,y'' = x2\ -\ x\ y\ .$

So the iteration f that we apply to the non-negative solution pairs (x,y) of the relation x2 + y2 - 2kxy - x = 0 works as follows:

- if $0 \leq y < x$ then $f = V \circ H$; the output is the pair (x',y');

- if $1 \leq x < y$ then $f = H \circ V$; the output is the pair (x'',y'');
- if both x and y are 0, then the output is (0,0).

So we have at hand a valid iteration scheme. The crucial point is that given any solution pair (x,y) with $x > 1$ (note that this condition forces y to be positive), we can apply f and get another solution pair (x1,y1) with $0 < x1 < x$. This step can be repeated, provided that $x1 > 1$. The iteration may be continued in this manner, and we get a sequence of steadily decreasing positive integers. Clearly, this cannot be continued indefinitely; the iteration must come to a halt at some point. This will happen when we reach a pair of the form (1, yk) for some positive integer yk. But we have

$$(1,yk) \to V\ (1,0) \to H(0,0).$$

Therefore, to any non-negative solution (x,y) *of the relation* $x2 + y2 - 2kxy - x = 0$, *we can apply some sequence of* H' *s and* V *'s which will take us to the pair* (0, 0).

Now we simply back-track—by applying the reverse sequence of V 's and H's to the pair (0,0); we get

$$(0,0) \to H(1,0) \to V(1,2k) \to H(4k2,2k) \to V\cdots.$$

Observe that the first member of each pair is a square. This will always be the case; for if x is a square and $x|y2$, then $y2/x$ too is a square.

The reverse sequence must reach the starting pair (a,b) at some point, and this means that a is a square. Our analysis has, incidentally, found a way of enumerating all solutions of the given equation. ❑

17.5 Exercises

17.5.1 Prove Theorem 17.3.

17.5.2 Prove Theorem 17.4.

17.5.3 Prove Theorem 17.5.

17.5.4 Prove Theorem 17.6.

Chapter 18

Miscellaneous Problems

In conclusion, we consider a few challenging iteration problems which have appeared in the Olympiads and the problems sections of journals such as CRUX MATHEMATICORUM (published by the Canadian Mathematical Society, CRUX for short; we note in passing that CRUX has a very active and popular Olympiad Corner featuring problems from Olympiads the world over.) In a few of the problems there may seem to no iteration involved; but that is by design: the problems have been posed in such a way as to mask the connection! (For olympiad problem posers, this is of course a well known stratagem!)

The approaches used to solve the problems encompass a lot of diversity and possess considerable instructional value; we urge the reader to make a close study of the techniques used.

Please try to solve the problems on your own!

Problem 1 (Spanish Mathematical Olympiad, 1973)

Find the real numbers x1,x2,…,xn satisfying the following equations:

$$x12 + 3x\,1 + 1 = x2,$$

$$x22 + 3x\,2 + 1 = x3,$$

$$x32 + 3x\,3 + 1 = x4,$$

$$\cdots\cdots\cdots = \cdots$$

$$xn\text{-}12 + 3x\,n\text{-}1 + 1 = xn,$$

$$xn2 + 3x\, n + 1 = x1.$$

Problem 2

Find all triples (x,y,z) of real numbers such that y = 4x3 - 3x,z = 4y3 - 3y and x = 4z3 - 3z.

Problem 3 (from CRUX)

Let functions f,f1,f2,… be defined on N as follows: f(n) = (SUM(n))2 is the square of the sum of the digits of n (written in base-10), and for integers $k \geq 1$,fk(n) is the k-fold iterate of f acting on n; i.e.,

$$fk(n) = f(k)(n) = \underbrace{f(f(f(\cdots f(n))))}_{k\text{times}}.$$

For example, f1(25) = 49,f2(25) = f(49) = 169, and so on.

Problem. Compute f1991(21990). (The problem is from CRUX.)

Problem 4

Let f be a function from **R** into **R** defined as follows: f(x) = x2- 2. Show that there is no function g from **R** into **R** such that $g \circ g = f$; in other words, that there is no real-valued function g such that g(g(x)) = x2 - 2 for all $x \in \mathbf{R}$.

This problem was posed in Chapter 10 (Problem 10.5.4).

Problem 5 (Irish Mathematical Olympiad, 1991)

Three operations a,b and c are defined as follows:

a(n) = 10n (for all $n \in N$);

b(n) = 10n + 4 (for all $n \in N$);

c(n) = n/2 (for even $n \in N$).

Prove that starting from 4, every natural number can be obtained by performing a finite number of the operations a,b and c in some order.

Problem 6 (Putnam Competition, December 2000)

Let f(x) be a polynomial with integer coefficients. Suppose that f has a cycle C containing an integer entry. Show that the length of C is either 1 or 2.

You may now turn the page for the solutions!

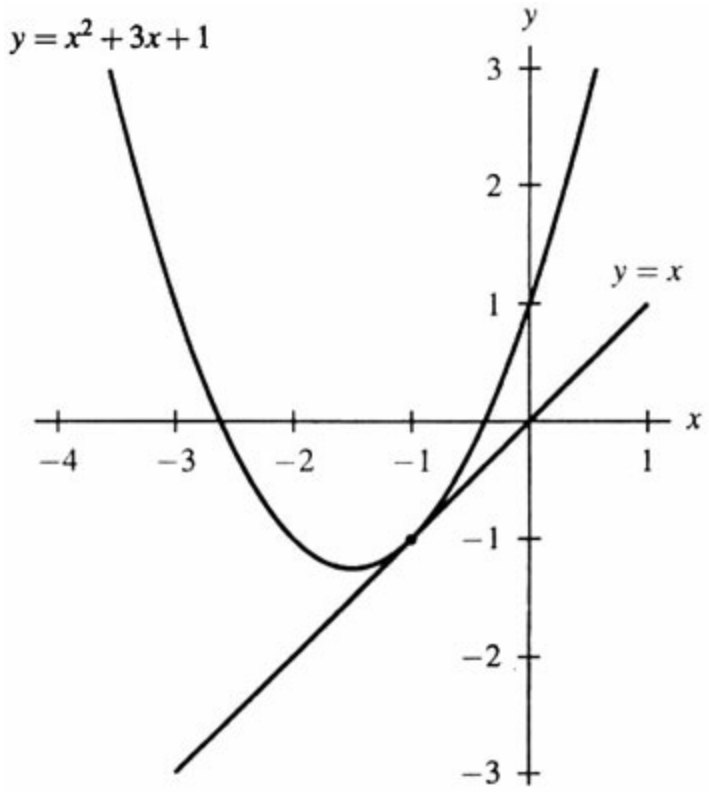


Figure 18.1. *Graphs of* y = x2 + 3x + 1 *and* y = x

Solution of Problem 1

We shall show that this is essentially an iteration problem in disguise and solve it accordingly. Write f for the following function which acts on R:

$$f(x) = x2 + 3x + 1.$$

Then the given system is equivalent to:

$$x2 = f(x1), x3 = f(x2), \ldots, xn = f(x1), x1 = f(xn).$$

So the problem is to find the n-cycles of f, for any given positive integer n. Now observe that

$$f(x) = (x2 + 2x + 1) + x = (x + 1)2 + x \geq x.$$

The inequality sign holds because (x + 1)2 is a square and so is non-negative.

(See Figure 18.1 for the graphical expression of this inequality.) Equality holds if and only if x + 1 = 0, i.e., iff x = -1. It follows from this that

$$f(x) \geq x,$$

$$f(f(x)) \geq f(x) \geq x,$$

$$f(f(f(x))) \geq f(f(x)) \geq f(x) \geq x,\ldots,$$

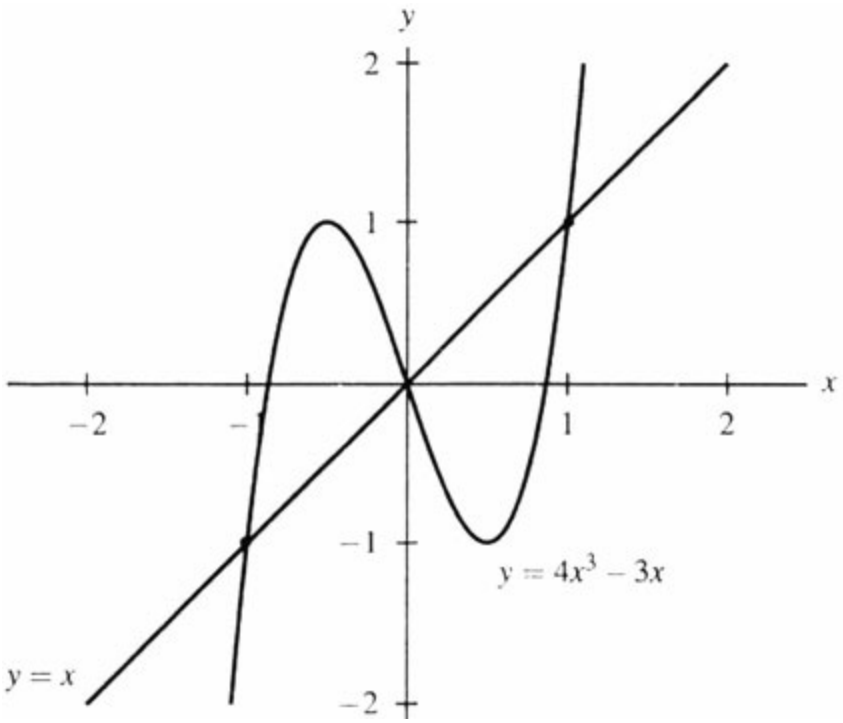


Figure 18.2. *Graphs of* y = 4x3 - 3x *and* y = x

and the equality sign holds in this chain of inequalities if and only if x = -1. So by iterating f on any starting number x≠ - 1, we will have

$$x < f(x) \leq f(f(x)) \leq f(f(f(x))) \leq f(f(f(f(x)))) \leq \cdots,$$

implying that if x≠ - 1, then x < fn(x) for every n ∈ **N**. Therefore if x = fn(x), then it must be that x = -1, in which case x = fn(x) for every n.

So f has precisely one 1-cycle, ⟨-1⟩, and no higher order cycles. Therefore for any given n ≥ 1, the only solution to the given system of equations is

$$x1 = x2 = \cdots = xn\text{-}1 = xn = -1.$$

Solution to Problem 2

Let P(x) denote the polynomial 4x3 - 3x. The question asks in effect for the 1-cycles and 3-cycles of P. The 1-cycles are easily found; the equation x = P(x) or x = 4x3 - 3x yields x = 0.1,-1, so there are precisely three 1-cycles.

To find the 3-cycles, note that if x > 1 then 4x3 > 4x, ∴P(x) > x; and if x < -1 then 4x3 < 4x, ∴P(x) < x. Moreover, P maps the interval I = [-1,1] on itself. These claims are easy to verify using the graph of P (see Figure 18.2), so we do not give any formal proofs. From them we infer that the cycles of P (if any) are entirely contained in the set I.

We now draw on the trigonometric formula

$$\cos 3t = 4\cos 3t - 3\cos t,$$

and the fact that cosx decreases steadily from 1 to −1 as x increases from 0 to π. For any number x ∈ [-1,1], there exists a unique number t in the interval J = [0,π] such that x = cost. Supposing that x ∈ I is part of a 3-cycle, we write x = cost where t ∈ J; then we have

$$P(x) = \cos 3t, P \circ P(x) = \cos 9t, P \circ P \circ P(x) = \cos 27t.$$

Since x belongs to a 3-cycle, we have cos27t = cost. All we need to do now is to solve this equation over J! The only possibilities are:

- 27t - t = 0,2π,4π,6π,…; i.e., 26t is an even multiple of 2π, so:

$$t = 0, \pi\ 13, 2\pi\ 13, 3\pi\ 13, \cdots, 12\pi\ 13, \pi;$$

- 27t + t = 0,2π,4π,6π,…; i.e., 28t is an even multiple of 2π, so:

$$t = 0, \pi\ 14, 2\pi\ 14, 3\pi\ 14, \ldots, 13\pi\ 14, \pi.$$

The desired x-values are simply the cosines of these numbers. Since cosθ = cos(2π - θ) for any θ, the distinct values of x so obtained are, respectively,

$$x = \cos 0, \cos \pi\ 13, \cos 2\pi\ 13, \ldots, \cos 12\pi\ 13, \cos 13\pi\ 13,$$

$$x = \cos 0, \cos \pi\ 14, \cos 2\pi\ 14, \ldots, \cos 13\pi\ 14, \cos 14\pi\ 14.$$

The 1-cycles correspond to t = 0,π/2 and π (giving x = 1,0 and − 1 respectively), and the other t-values yield 3-cycles. So the 3-cycles are all of the form ⟨x,y,z⟩, where x = cosnπ/13 with n = 1,2,3,…,12, or x = cosnπ/14 with n = 1,2,3,...,...,13, and y = P(x),z = P(y). There are 24 x-values, and they

yield eight 3-cycles (they obviously get grouped into sets of three each).

For example, $x = \cos\pi/13$ yields the 3-cycle

$$\cos \pi\ 13, \cos 3\pi\ 13, \cos 9\pi\ 13 \approx \langle 0.970942, 0.748511, -0.568065 \rangle,$$

and $x = \cos 3\pi/14$ yields the 3-cycle

$$\cos 3\pi\ 14, \cos 9\pi\ 14, \cos 27\pi\ 14 \approx \langle 0.781831, -0.433884, 0.974928 \rangle.$$

Solution of Problem 3

We shall solve the problem by finding the cycle structure of f.

Let $a_n = f^n(21990)$; then $a_0 = 21990$. We are required to compute a_{1991}. Observe that since $\mathrm{SUM}(k) \equiv k \pmod 9$ for any integer k, we have

$$f(k) \equiv k^2 \pmod 9 \text{ for all integers } k \geq 1.$$

From this it follows that for all $n \geq 1$,

$$a_{n+1} \equiv a_n^2 \pmod 9.$$

Next, note that, modulo 9, we have:

$$2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 8, 2^4 \equiv 7, 2^5 \equiv 5, 2^6 \equiv 1,$$

$$2^7 \equiv 2, 2^8 \equiv 4, 2^9 \equiv 8, 2^{10} \equiv 7, 2^{11} \equiv 5, 2^{12} \equiv 1,$$

and so on. So the remainders of the powers of 2, listed modulo 9, form the 6-cycle $\langle 2,4,8,7,5,1 \rangle$. Since 1990 is of the form 4 (mod 6), we get:

$$2^{1990} \equiv 7 \pmod 9.$$

So $a_0 \equiv 7 \equiv -2 \pmod 9$. Since $a_{n+1} \equiv a_n^2 \pmod 9$ we have, modulo 9:

$$a_1 \equiv 4, a_2 \equiv 7, a_3 \equiv 4, a_4 \equiv 7,$$

$$a_5 \equiv 4, a_6 \equiv 7, a_7 \equiv 4, a_8 \equiv 7, \ldots.$$

The pattern is immediately visible; we see that for $n \geq 1$,

$$a_n \equiv 4 \pmod 9 \text{ if } n \text{ is odd}, 7 \pmod 9 \text{ if } n \text{ is even}.$$

Now we observe the following inequalities:

$$a_0 = 21990 < 10^{600},$$

$\therefore a1 < (9 \times 600)2 < 108,$

$\therefore a2 < (9 \times 8)62 < 104,$

$\therefore a3 < (9 \times 4)2 = 1296,$

$\therefore a4 < (9 \times 3)2 = 729.$

We conclude from this that an < 729 for all n ≥ 4.

The largest SUM for integers not exceeding 729 occurs for the integer 699, whose SUM is 24, so:

$$SUM(an) < 24 \text{ for all } n \geq 4.$$

This implies that an < 576 for all n ≥ 5. The largest SUM for integers not exceeding 576 occurs for the integer 499, whose SUM is 22, so:

$$SUM\ (an) < 22 \text{ for all } n \geq 5.$$

This implies that an < 484 for all n ≥ 6. The largest SUM for integers not exceeding 484 occurs for the integer 399, whose SUM is 21, so:

$$SUM\ (an) < 21 \text{ for all } n \geq 6.$$

It follows that an < 441 for all n ≥ 7.

We have indeed achieved a huge reduction in size: starting from 21990 (which has six hundred digits!), we have reached numbers not exceeding 441. It should be clear that once we reach a number below 441, we shall never go above that figure. So the set of integers below 441 is a "black hole set" or a "basin of attraction" for the function f.

In fact, a whole lot more can be said. We showed above that an ≡ 4 (mod 9) if n is odd, and an ≡ 7 (mod 9) if n is even. We deduce from this that

$$SUM\ (an) \equiv 4(\text{mod}9)\text{ifnisodd, }7(\text{mod }9)\text{ if nis even .}$$

Since further we know that SUM(an) < 21 for all n ≥ 6, we see that for n ≥ 6, if n is odd, then SUM (n) = 4 or 13, and if n is even, then SUM(n) = 7 or 16. Since an+1 = (SUM(an)) 2 we further get:

$$\text{ifn} \geq 7\text{iseven,an} = 16\text{or}169,\ \text{ifn} \geq 7\text{isodd,an} = 49\text{or}256.$$

Finally, since (4 + 9)2 = 169 and (2 + 5 + 6)2 = 169, we get:

$$\text{if} n \geq 8 \text{is even}, a_n = 169. \quad \text{if} n \geq 8 \text{is odd}, a_n = 256.$$

This means that f has precisely one cycle!—the 2-cycle $\langle 169,256 \rangle$. Since every orbit must fall into this cycle sooner or later, it follows that $f^{1991}(21990) = 256$.

Solution to Problem 4

We start by making a few general observations. Our approach is much the same as the one used to solve the IMO 1987 problem in the preceding chapter.

Let $f : R \to R$ be any function. Consider the relation between the cycles of f and those of $f \circ f$. As pointed out in Chapter 17, under action by f, each even cycle C of f splits into two subsets of equal cardinality, say C1 and C2, each of which is a cycle of $f \circ f$ (moreover, C1 and C2 consist of alternate elements of C). On the other hand, an odd cycle C remains intact under action by f; for, by selecting the alternate elements of the cycle, we end up selecting all the elements. So C is a cycle of $f \circ f$ as well as of f.

Moreover, every cycle of $f \circ f$ arises in this manner. We deduce from this the crucial finding that every even cycle of $f \circ f$ has a mate, namely another cycle with the same number of elements, such that the two can be woven together to give a cycle of f. So for each fixed even number k, the number of cycles of $f \circ f$ with k elements must be an even number. Accordingly, we arrive at the following general principle.

> *If for a given function* $g : R \to R$ *and some even integer* k , *it is the case that the number of k-cycles of* g *is odd, then there cannot exist any function* $f : R \to R$ *such that* $g = f \circ f$.

We apply this general principle to the problem at hand. Let $g(x) = x^2 - 2$. Solving the equation $x = g(x)$ we find the 1-cycles of g to be $\langle -1 \rangle$ and $\langle 2 \rangle$. For the 2-cycles we examine the equation $x = (x^2 - 2)^2 - 2$. The equation simplifies to

$$x^4 - 4x^2 - x + 2 = 0, \therefore (x - 2)(x + 1)(x^2 + x - 1) = 0.$$

The solution set of this is $\{-1,2,\alpha,-1/\alpha\}$, where

$$\alpha = \frac{5 - 1}{2}.$$

We have already seen that - 1 and 2 correspond to the 1-cycles of g. (Observe how the odd cycles of g ◦ g correspond to the odd cycles of g.) On the other hand,

$$g(\alpha) = \alpha 2 - 2 = -\alpha - 1 = -1\ \alpha,$$

$$g\ -1\ \alpha = 1\ \alpha 2 - 2 = 1\ \alpha - 1 = \alpha,$$

so $\langle \alpha, -1/\alpha \rangle$ is a genuine 2-cycle of g; and it is the *only* 2-cycle; therefore, the number of 2-cycles is odd. Invoking the general principle proved above, we see that there does not exist any function f from R into itself such that f ◦ f = g. This proves the assertion.

Remark. Note that this is a parity based argument. Simple considerations can yield far-reaching results!

Solution to Problem 5

We turn the problem "upside down" and formulate it as a "Collatz-type" problem, as follows.

Let the function f *be defined on* $\mathrm{N}0 = \mathrm{N} \cup \{0\}$ *thus*:

$$f(n) = n/10 \text{ if n is a multiple of 10; } (n - 4)/10 \text{if n is of the from 4(mod 10); } 2n \text{ else.}$$

Show that for any integer n, *the orbit of* n *under iteration by* f *contains the number* 4.

Examples of f-orbits are:

- $11 \mapsto 22 \mapsto 44 \mapsto 4 \mapsto \cdots$;
- $13 \mapsto 26 \mapsto 52 \mapsto 104 \mapsto 10 \mapsto 1 \mapsto 2 \mapsto 4 \mapsto \cdots$;
- $18 \mapsto 36 \mapsto 72 \mapsto 144 \mapsto 14 \mapsto 1 \mapsto 2 \mapsto 4 \mapsto \cdots$.

Observe that we have "hit" 4 in each case.

Strictly speaking, this problem is not logically equivalent to the original one, but it does *imply* the desired result. (If we reverse the path taken by any number n to 4, we get the desired route.)

We first claim that the only fixed point of f is 0. To see why, observe that if n is a non-zero multiple of 10 or of the form 4 (mod 10), then f(n) < n; and if n is non-zero and not a multiple of 10, nor of the form 4 (mod10), then f(n) > n.

Since f(4) = 0, and 0 is the only fixed point of f, we may further reformulate the assertion to be proved thus: *The orbit of any positive integer n under action by* f *ultimately reaches the 1-cycle* ⟨0⟩. This is the assertion that we shall prove.

Table 18.1 displays, for n ≡ 0,1,2,…,8 and 9 (mod 10), a relevant member of the orbit of n, along with its parity; we have written fr for the r-fold iterate of f, with f1 = f.

The crucial observation that can be made from the table is this: *If* n ≡ 0,1,2, ..., 7 *or* 8 (mod 10), *then in* 4 *steps at most we will* reach *an even number which is strictly smaller than* 4n⁄5.

If n ≡ 9(mod10), however, the conclusion does not hold; we have f5(n) = (8n - 2)⁄5, which exceeds n. However note that f5(n) is *even*. So the following holds: *If n is of the form* 9 (mod 10), *then in* 5 *steps we reach an even number* n1 *which is strictly smaller than* 8n⁄5. We shall use this observation to proceed.

Clearly, n1 ≡ 0,2,4,6 or 8 (mod 10). We consider the various possibilities.

- If n1 ≡ 0 or 4 (mod 10), then in 1 more step we reach a number n2 for which the following holds: n2 ≤ n⁄10 < 8n⁄50 < n.

Form of n	Relevant iterate	Parity
$n = 10k$	$f_1(n) = n/10 = k$	?
$n = 10k+1$	$f_3(n) = (4n-2)/5 = 4k$	even
$n = 10k+2$	$f_2(n) = (4n-4)/5 = 8k$	even
$n = 10k+3$	$f_4(n) = (4n-2)/5 = 8k+2$	even
$n = 10k+4$	$f_1(n) = (n-4)/10 = k$	?
$n = 10k+5$	$f_2(n) = n/5 = 2k+1$	odd
$n = 10k+6$	$f_3(n) = (2n-2)/5 = 4k+2$	even
$n = 10k+7$	$f_2(n) = (2n-4)/5 = 4k+2$	even
$n = 10k+8$	$f_4(n) = (4n-2)/5 = 8k+6$	even
$n = 10k+9$	$f_5(n) = (8n-2)/5 = 16k+14$	even

Table 18.1. *Tracing the path taken by* n

- If n1 ≡ 2,6 or 8 (mod10), then in 4 more steps at most we reach an even number n2 for which the following holds: n2 < 4n1⁄5 < 32n⁄25. (This follows from the observation made above.)

In the latter case we cannot be sure that n2 is less than n; more iterations are required, so we consider the different forms that n2 can take. We have now:

- If n2 ≡ 0 or 4 (mod 10), then in 1 more step we reach a number n3 for which the following holds: n3 ≤ n2⁄10 < 32n⁄250 < n.
- If n2 ≡ 2,6 or 8 (mod 10), then in 4 more steps at most we reach an even number n3 for which the following holds: n3 < 4n2⁄5 < 128n⁄125.

In the latter case we continue to be "above" n; so we consider the different forms that n3 can take. We have:

- If n3 ≡ 0 or 4 (mod 10), then in 1 more step we reach a number n4 for which the following holds: n4 ≤ n3⁄10 < 128n⁄1250 < n.
- If n3 ≡ 2,6 or 8 (mod 10), then in 4 more steps at most we reach an even number n4 for which the following holds: n4 < 512n⁄625 < n.

So for all n, *in at most* $5 + 4 + 4 + 4 = 17$ *steps we reach a number* n' *which is strictly smaller than* n. This allows for a neat finish to the analysis. We use the following lemma (in the statement, N0 means $\mathbf{N} \cup \{0\}$).

Lemma 18.1 *Let* g *be a mapping from* N0 *into* N0, *with* $g(0) = 0$. *Suppose that there exists a positive integer* k *such that for every* n *for which* $g(n)$ *is positive, we have* $g(n + i) < g(n)$ *for some* i *in the set* $\{1, 2, \ldots, k\}$; *then, under g-iteration, every orbit ultimately reaches* 0.

The proof is not hard. Steady reduction occurs every k steps at most, so, starting at n where $g(n) > 0$, in at most $kg(n)$ steps we must hit 0.

The function f in our problem has all the features required by the lemma, with $k = 17$. The result follows: *every orbit reaches* 0 *at some stage*.

Solution of Problem 6

The claim that for a polynomial f(x) with integer coefficients, the length of a f-cycle C containing an integer entry can only be 1 or 2 looks surprising, but it can be proved rather easily. We only need the following result: *If* f(x) *is a polynomial with integer coefficients, and* a *and* b *are unequal integers, then* a - b *is a divisor of* f(a) - f(b). This is so because a - b is a divisor of an - bn for every positive integer n.

Let C be f-cycle containing an integer entry; then the entries of f are all integers. Let the cycle be $\langle a1,a2,a3,\ldots,ak\rangle$, for some integer $k \geq 1$. Of course, the ai's are unequal. By definition, we have

$$f(a1) = a2, f(a2) = a3, \cdots, f(ak-1) = ak, f(ak) = a1.$$

We shall suppose that $k \geq 3$, and derive a contradiction.

Let $b1 = a2 - a1, b2 = a3 - a2, \cdots, bk-1 = ak - ak-1, bk = a1 - ak$. Then the bi are all non-zero integers. Using the divisibility result quoted earlier we get

$$b1|b2, b2|b3, bk-1|bk, bk|b1,$$

and therefore,

$$|b1| \leq |b2| \leq |b3| \leq \cdots \leq |bk| \leq |b1|.$$

Since the initial and last terms in this string of inequalities are identical, the

various ‘ ≤’ signs must all become equalities; so we get

$$|b1| = |b2| = |b3| = \cdots = |bk|.$$

So we have:

$$|a2 - a1| = |a3 - a1| = |a4 - a3| = \cdots = |ak - ak\text{-}1| = |a1 - ak|.$$

Since a1,a2,a3,⋯,ak are all unequal, this implies that

$$a2 - a1 = a3 - a2 = a4 - a3 = \cdots = ak - ak\text{-}1 = a1 - ak.$$

Let a2 - a1 = d; then we get a2 = a1 + d,a3 = a1 + 2d, and so on; and finally, ak = a1 + (k - 1)d, which means that

$$a1 - ak = -(k - 1)d, \therefore d = -(k - 1)d.$$

But this equation is clearly impossible. So we cannot have k ≥ 3. It follows that k = 1 or 2.

Note that k = 1 or 2 does not give rise to any contradiction. The former merely means that f has an integral fixed point, and this is clearly possible. The latter means that we have

$$f(a1) = a2, f(a2) = a1.$$

There is no difficulty with this either. For example, we can have f(x) = -x, for which ⟨a,-a⟩ is a 2-cycle for any real non-zero number a, and in particular for any positive integer a. ❑

Appendix A

Further Reading

There are unfortunately relatively few books on the topic of iteration which are suitable for students at the high-school level. The reader will notice that many of the references are for the topic of fractals rather than on iteration *per se*. (Some of the pieces are from journals rather than books.)

[DAVIS] Davis, C and Knuth, D E. *Number representations and dragon curves*, in JOURNAL OF RECREATIONAL MATHEMATICS, Volume 3, pages 66–81 and 133–149.

[DUNHAM] Dunham, William. JOURNEY THROUGH GENIUS (Mathematical Association of America).

[GLEICK] Gleick, James. CHAOS: MAKING A NEW SCIENCE (Viking, New York, 1987).
This was a best seller in its time, and presents a popular account of the manner in which the theory of chaos developed; highly recommended.

[HOFSTADTER] Hofstadter, Douglas. *Strange Attractors*: *mathematical patterns delicately poised between order and chaos*, in SCIENTIFIC AMERICAN, Volume 245 (November), pages 281–291.

[MANDELBROT] Mandelbrot, Benoit. THE FRACTAL GEOMETRY OF NATURE (W H Freeman, 1982).
This is a very famous and much acclaimed book, and it presents a new way of viewing nature, beginning with the much-quoted words "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line".

[SMITAL] Smital, Jaroslav. FUNCTIONS AND FUNCTIONAL EQUATIONS (Poland).
This is a more advanced text, but is extremely well-written and informative.

[VILENKIN] Vilenkin, N Ya. THE METHOD OF SUCCESSIVE APPROXIMATIONS (MIR Publishers, Moscow).
This is part of the 'Little Mathematics Library' (a series of books published in the former Soviet Union) and is an excellent buy. It discusses in detail the theme introduced in Chapter 15: that of solving equations using the iteration. In particular, it answers the question asked in the chapter, namely: *When does the technique of iteration work, and when does it fail* ?

Appendix B

Solutions

3.1.1 Yes; c < b, because the remainder is always less than the quotient.

3.1.2 The orbits are:

i. (100, 63), (63, 37), (37, 26), (26, 11), (11,4), (4, 3), (3, 1), (1,0). The gcd is 1.

ii. (100, 78), (78, 22), (22, 12), (12, 10), (10, 2), (2,0). The gcd is 2.

iii. (100, 17), (17, 15), (15, 2), (2, 1), (1, 0). The gcd is 1.

3.2.1 The output after n repetitions of the operation is x2n

3.2.3 We use the following result which was proved in the text: *if* x > 1, *then* x2n *tends to infinity with* n.

If 0 < x < 1, then 1⁄x > 1 and we apply this result to the quantity (1⁄x)2n = 1⁄n; this will tend to infinity with n, so x2n must tend to 0. It follows that when 0 < x < 1 then the quantity x2n tends to 0 as n tends to infinity. If–1 < x < 0, then x2 lies between 0 and 1, and the same argument applies from this point on; the limit is 0 in this case too.

3.2.4 The output after the procedure has been repeated n times is 102n , so we need to find the least n for which 102n > 10 1000. The answer is n = 10, because 210 = 1024 > 1000 and 29 = 512 < 1000. Similarly, 102n > 1010000 when n = 144 because 214 = 16384, whereas 213 = 8192 < 1000.

3.2.5 As earlier, we need to find the least value of n for which 22n > 101000. To solve this inequality for n, note that

$$2212 = 24096 > 24000 = 161000 > 101000,$$

while

$$2211 = 22048 < 23000 = 81000 < 101000$$

This means that the answer is n = 12.

Similarly, to find the least n for which 22n > 1010000, we note that

$$2216 = 265536 > 260000 = 6410000 > 1010000,$$

while

$$2215 = 232768 < 233000 = (233)1000 < (1010)1000 = 1010000,$$

because 233 = (211)3 = 20483 < 21003 = 9261000000 < 1010; therefore the answer is n = 16.

3.2.6 The output after n iterations is x3n

3.3.1 The limit is 1. To prove this is a bit tricky, though; to do so, we shall use the fact that for a > 0,

$$1 < 1 + a < 1 + a\,2.$$

(The first inequality is obvious, and the second one becomes clear after squaring both sides, for 1 + a < 1 + a + a2⁄4.) Let the initial number be written as 1 + a, where a >0. Then the number obtained after n extraction of the square root is

$$(1 + a)1⁄2n\ (= an, \text{say}).$$

Invoking the inequality 1 < 1 + a < 1 + a⁄2 repeatedly, we obtain

$$1 < an < 1 + a\,2n.$$

As n increases without limit, the quantity a⁄2n shrinks to 0 very rapidly, so 1 + a⁄2n converges to 1. This shows that the limit of anis 1.

The argument for the case a <0 is similar, and the details are left to the reader.

3.3.2 The argument in this case is very similar; the limit is 1, no matter what input is chosen. The proof uses the inequality.

$$1 < 1 + a3 < 1 + a\,3$$

which holds for all a > 0.

3.3.3 The function x →x is clearly the inverse of the functionx→x2 ; thus the sequences produced by the two functions are reversals of one another.

4.2.1 The trajectories are as displayed below.

(i)	(1,17,63,6), (19,41,41,19) (0,0,0,0).	(16.46,57.5), (22.0.22.0).	(30.11.52.11), (22.22.22.22).
(ii)	(91,117,13,60), (21,41,11,73), (22,22,22,22),	(26.104.47.31). (20.30.62.52), (0.0.0.0).	(78.57.16.5). (10.32.10.32).
(iii)	(236,40,67,3), (132,132,132,132),	(196,27,64,233), (0,0,0,0).	(169,37,169,37)
(iv)	(47,3,102,200), (54,54,54,54),	(44,99,98,153), (0,0,0,0).	(55,1,55,109)

4.2.2 The quadruples that reach (0,0,0,0) after just one round of the iteration are all of the form (k,k,k,k) for some number k; those that take two rounds to reach (0,0,0,0) are all of the form (a,b,a,b), for some numbers a,b(a≠b); and those that reach (0,0,0,0) after three rounds are of the form (k,k + a,k + b,k + |a - b|), where a,b,k are any numbers with b≠0,2a. For example, with k = 1,a = 3,b = 4, we obtain the quadruple (1,4,5,2), whose orbit is:

$$(1,4,5,2)\mapsto(3,1,3,1)\mapsto(2,2,2,2)\mapsto(0,0,0,0).$$

(You should try to justify these claims on your own.)

4.2.3 Line 80 evaluates the sum of the elements of the quadruple, and line 90 checks if the quadruple (0,0,0,0) has been reached, because if s > 0, then some number of the quadruple is non-zero, so more iterations are needed. (Remember that after the first round, the quadruple has no negative numbers, so there.is no danger of zero arising as a result of cancellation.)

4.2.4 The trajectories are as displayed below.

i. (2,13,17),(11,4,15),(7,11,4),(4,7,3),(3,4,1),(1,3,2),(2,1,1),(1,0,1),(1,1,0), (0,1,1),(1,0,1),⋯. We have reached a 3-cycle.

ii. (3,19,50),(16,31,47),(15,16,31),(1,15,16),(14,1,15),(13,14,1),(1,13,12),

(12,1,11),⋯. Ultimately we reach a 3-cycle.

iii. (100,201,18),(101,183,82),(82,101,19),(19,82,63),(63,19,44),(44,25,19), (19,6,25),(13,19,6),(6,13,7),(7,6,1),(1,5,6),⋯. Once again a 3-cycle!

4.2.5 The triples that ultimately reach (0,0,0) are of the form (a,a,a) for some number a.

If a,b,c are not all equal, let k denote the gcd of the three numbers |a - b|,|b - c|,|c - a|. Then the orbit eventually reaches the 3-cycle

$$(k,k,0)\mapsto(0,k,k)\mapsto(k,0,k)\mapsto\ldots.$$

Note that this also holds if two out of a,b,c are equal. For example, if a = 10,b = 10,c = 98, then the three differences are 88, 88 and 0, whose gcd is 88. Thus the orbit reaches the triple (88,88,0) and then stays in a 3-cycle.

4.2.6 You will find that a law similar to that stated above holds. If the initial 5-tuple is. (a,b,c,d,e), and the numbers a,b,c,d,e are not all equal, let k denote the gcd of the five numbers |a - b|,|b - c|,|c - d|,|d - e| and |e - a|. Then the orbit eventually reaches a tuple composed only of k's and 0's, after which it stays in a cycle.

A similar law holds for 6-tuples.

5.2.1 The orbits are:

i. ⟨ 5,25,29,85,89,145,42,20,4,16,37,58,89,…⟩ .

ii. ⟨ 6,36,45,41,17,50,25,29,85,89,…⟩ .

iii. ⟨ 8,64,52,29,85,89,…⟩ .

iv. ⟨ 11,2,4,16,37,58,89,…⟩ .

v. ⟨ 12,5,25,29,85,89,…⟩ .

Note how the same cycle is reached in each case.

5.2.2 Listed below is a simple computer program in *BASIC* that will capture all seeds that lie between two specified limits (nmin, nmax) and that finally reach the cycle ⟨ 1⟩ .

```
CLS
INPUT nmin, nmax
REM nmin and nmax should be less than 1000
FOR number = nmin TO, nmax
   n = number
   WHILE n > 1 AND n <> 20
      a = n MOD 10: n1 = (n -a) / 10
      b = n1 MOD 10: n2 = (n1- b) /10
      c = n2 MOD 10
      n = a^ 2 + b^ 2 + c ^ 2
      IF n = 1 THEN PRINT number
   WEND
NEXT number
```

The condition " n <> 20" has been included to detect whether the orbit reaches the cycle containing 20, namely, ⟨20,4,16,37,58,…⟩.

On running the program with nmin = 1,nmax =100, we find that the seeds that eventually reach ⟨1⟩ are:

1,7,10,13,19,23,28,31,32,44,49,68,70,79,82,86,91,94 and 97.

The list is surprisingly large!

5.2.3 Obviously, the numbers that reach 1 after just one iteration are 1,10,100,1000,…; that is, the powers of 10. The numbers that reach 1 after two iterations are those that reach a positive power of 10 after one iteration. To list these down, we try to find all possible ways of expressing 10, 100,1000, … , as sums of squares of digits. For example: 10 = 12 + 12 + 22 + 22 = 12 + 32, and so on. Of course, we can include 02 as many times as we wish. This immediately shows that the numbers 1122, 2121, 2211, 13, 31, 103, 301, 10201020, …, all reach 10 after one iteration, and hence reach 1 after two iterations. It follows that there are infinitely many numbers that reach 1 after

two iterations! Similarly, we list down all the ways in which 100 can be written as a sum of squares of digits, for example 100 = 62 + 82, or

$$100 = 32 + 32 + \cdots + 32\overbrace{\quad} \text{ eleven times} + 1,$$

which shows that the numbers 68 and 333333333331 (with eleven 3's and one 1) reach 100 in one step and hence 1 in two steps. This procedure can be generalized in a fairly obvious way.

5.2.4 The only other numbers that have this property are: 1, 370, 371 and 407. How will you show that there are no four digit numbers with the property?

5.2.5 The other numbers that have this property are 1, 1634 and 9474. (Check: 14 + 64 + 34 + 44 = 1 + 1296 + 81 + 256 = 1634, as claimed; similarly for 9474.)

5.2.6 The detailed exposition in Chapter 14 (Section 14.1) may suggest a solution for this problem. Or we may proceed directly as follows.

If n is a k-digit number, then certainly SSQ (n) ≤ 81k. So we have:

$$\text{If } 10k\text{-}1 < k \leq 10k, \text{ then SSQ } (n) \leq 81k.$$

It is easily shown that for k ≥ 4 we have 81k < 10k-1. It follows that if n has four or more digits, then SSQ (n) < n. So there are no solutions to the equation n = SSQ (n) if n has four or more digits. So to check for solutions, we need to consider the cases when n has one, two or three digits.

If n has one or two digits, we write n = 10a + b where a and b are digits. (If a = 0 then n is a single digit number.) Then we have: n = SSQ (n), therefore

$$10a + b = a2 + b2, \therefore b2 - b = a(10 - a).$$

Since b2 - b is even for any integer b, it follows that a is even; i.e., a = 0,2,4,6 or 8. Therefore:

$$b2 - b \in \{0,16,24\}.$$

The values taken by b2 - b for b = 0,1,2,…,9 are

$$0, 0, 2, 6, 12, 20, 30, 42, 56, 72.$$

The numbers 16 and 24 do not feature anywhere in this list, so we get: a = 0,b2 - b = 0, ∴b = 0 or 1, ∴n = 0 or 1. So the only solutions with one or two

digits to the equation n = SSQ (n) are 0 and 1.

If n has three digits, then SSQ (n) ≤ 81 × 3 = 243, so if n = SSQ (n) then n ≤ 243. If n ≤ 243, then we get: SSQ (n) ≤ 81 × 2 = 162, so n ≤ 162. So the first digit of n must be 1.

Now write (n = 100 + 10b + c, where b,c are digits. We get: n = SSQ (n), so 100 + 10b + c = 1 + b2 + c2. This yields:

$$c2 - c = 99 + b(10 - b), \therefore c2 - c \geq 99.$$

But this is clearly not possible if c is a digit. So there is no three digit number n for which n = SSQ (n). The stated conclusion now follows.

6.1.1 The equation x2 - 2 = x can be written as (x - 2)(x + 1) = 0, so its solutions are 2 and –1. These are the fixed points of f.

6.1.2 The equation 2x - x2 = x simplifies to x2 = x, whose solutions are 0 and 1. The set of fixed points is therefore {0,1}.

6.1.3

i. x2 - 12 = x simplifies to (x - 4)(x + 3) = 0, so the set of fixed points is {-3,4}.

ii. x3 - 3x = x yields the equation x3 = 4x or, equivalently, x(x - 2)(x + 2) = 0. The set of fixed points is {-2,0,2}.

6.1.4 The important fact is that x2 ≥ 0 for all (real) values of x. Therefore x2 + 1 ≥ 1 for all x. If x < 1 then we have

$$x2 + 1 \geq 1 > x,$$

so x2 + 1 > x. If x = 1, then x2 + 1 = 2, so x2 + 1 > x. Finally, if x > 1, then x2 > x, so x2 + 1 > x. Therefore x2 + 1 > x in each case, so f has no fixed points. Here is another proof for the case when x ≥ 0. Consider the expression (x - 1)2 = (x2 + 1) - 2κ. Since a square is never negative, (x - 1)2 ≥ 0, that is, x2 + 1 > 2x. But 2x ≥ x (since x ≥ 0), so the inequality follows.

6.1.5 The proof mimics the one given above, and we leave the details to the reader.

6.2.1

1 19 = 0.052631578947368421‾052631578947368421‾…

1 21 = 0.047619‾047619‾047619‾] …

1 29 = 0.034482758620689655172413793 1‾034482758620689655172413793 1‾…

6.2.2 The required fraction is 41 100 + 1 100 57 99 = 4116 9900 = 343 825.

6.2.3 Here are two fractions of the required type.

1 41 = 0.02439‾02439‾02439‾…

1 17 = 0.0588235294117647‾0588235294117647‾…

6.2.4 The cycles are:

Powers of 7 : ⟨7,9,3,1⟩ (a cycle of length 4),
Powers of 8 : ⟨8,4,2,6⟩ (a cycle of length 4).

6.2.5 A units digit of 2 must necessarily be followed by a units digit of 4 as 2 × 2 = 4) ; next, 4 must be followed by 8 (because 4 × 2 = 8), 8 must be followed by 6 (because 8 × 2 has units digit 6), and finally, 6 must be followed by 2 (because 6 × 2 has units digit 2), thus closing the cycle.

6.2.6 We reason in the same manner: 3 is followed by 9 (3 × 3 = 9),9 is followed by 7(9 × 3 has last digit 7), 7 is followed by 1, and 1 is followed by 3, thus closing the cycle.

6.2.7 Argue in the following manner. Suppose that we are doing the division $a \div b$. If the division works out exactly at some point, then we obtain a decimal with infinitely many 0's, and the problem is solved. If not, then the remainder at each stage of the division is positive, so it is one of the numbers 1, 2, 3, …, $b - 1$. There are only $b - 1$ of these remainders, so after some point a remainder will be encountered that has already been met earlier. From this point on, the decimal must recur. If you work out an example, say the division 12 ÷ 7, you will see immediately how this logic solves the problem.

7.2.1 The orbit is: ⟨1039,9171,8532,6174,…⟩.

7.2.2 The orbits are, respectively:

i. ⟨8619,8172,7443,3996,6264,4176,6174…⟩;

ii. ⟨1234,3087,8532,6174,…⟩;

iii. ⟨1235,4086,8172,7443,3996,6264,4176,6174,…⟩;

iv. ⟨9000,8991,8092,8532,6174,…⟩;

v. ⟨8000,7992,7173,6354,3087,8532,6174,…⟩.

7.2.3 Let n have digits a,b,c,d where $a \geq b \geq c \geq d$. Then

$$K(n) = 999(a - d) + 90(b - c).$$

Now $a - d \leq 9$ and $b - c \leq 9$ (because a,b,c,d are digits), so we obtain

$$K(n) \leq 999 \cdot 9 + 90 \cdot 9 = 9801.$$

that is, $K(n) \leq 9801$ for all n. Thus 9801 serves as an upper bound for K(n). The bound is reached if and only if $a - d = b - c = 9$, that is, $a = b = 9, c = d = 0$. The numbers n for which K(n) attains its maximum value of 9801 are therefore 99, 909, 9009, 9090 and 9900.

7.2.4 Any number of the form $aaaa^-$ reaches 0000^- in one step, and it is clear that these numbers are.the only ones that reach 0000^- in one step.

To find the numbers that reach 0000^- after two or more iterations, we need to find n such that $K(n) = aaaa^-$ for some a. To make progress we use a result from divisibility theory: *a number is divisible by* 9 *if and only if the sum of its digits is divisible by* 9. This implies that if integers m,n have the same sum of digits, then m - n is divisible by 9, and in turn that K(n) is divisible by 9 for all n. On the other hand, $aaaa^-$ is divisible by 9 iff $4a$ (its sum of digits) is divisible by 9, and this holds only for $a = 9$. But the equation $K(n) = 9999$ can never hold, because $K(n) \leq 9801$ for all 4-digit numbers n. So we reach a curious result: the numbers that ultimately reach 0000^- do so after just one iteration, and they are precisely the numbers of the form $aaaa^-$ for some digit a.

7.2.5 The only fixed points for the 4-digit Kaprekar iteration are 0000 and 6174.

7.2.6, 7.2.7 The only fixed points for the 3-digit Kaprekar iteration are 000 and 495. (The proof of this assertion has been discussed in outline in the text.)

7.2.8 The 5-digit Kaprekar iteration behaves quite differently from the 4-digit Kaprekar iteration—we find 2-cycles as well as 4-cycles:

- $\langle 59994,53955 \rangle$;
- $\langle 61974,82962,75933,63954 \rangle$;
- $\langle 62964,71973,83952,74943 \rangle$.

There appears to be no fixed point (other than 00000). Probably many other interesting facts can be uncovered. (The cycles listed above were found using a computer.)

In the case of the 6-digit Kaprekar iteration, we even find a 7-cycle! The cycle is:

$$\langle 840852,860832,862632,642654,420876,851742 \rangle.$$

7.2.9 In this case what happens is that occasionally the iteration comes to a quick halt. For instance, with a seed of 7776, we obtain K(7776) = 7776 - 6777 = 999. With no padding we would have K(K(7776)) = 0, and our investigation would now run out of material to investigate!

Note that this will happen only very occasionally: for four-digit numbers n, the iterate K(n) is a 3-digit number only when the digits of n are a,a,a,a ± 1 in some order (for some a), and it so happens that for all such n we have K(n) = 999. So the following statement can be made: the set of seeds n for which the two iterations give different results is

$$\{n : 1000 \leq n \leq 9999 : \text{some iterate of n equals } 999\}.$$

It is not too hard to enumerate this set. Remembering that K(n) is a multiple of 9 for all n, we see that we must first enumerate the digit a for which 4a ± 1 is a multiple of 9, i.e., for which $4a \equiv \pm 1 \pmod 9$. Multiplying both sides of the congruence by - 2 we obtain $a \equiv \pm 2 \pmod 9$ or $a \in \{2,7\}$. Continuing, we find that the only n for which (K(K(n)) = 999 is the set

$$\{2223,2232,2322,3222,6777,7677,7767,7776\}.$$

We must now enumerate the n's for which K(n) is one of *these* eight numbers. A computer search however fails to produce any such n, so we conclude that the set of n for which the two iterations (with and without

padding) behave differently is just the 8-element set listed above.

8.1.1 This has been proved in the text.

8.2.1

i. g(48) = 60

ii. g(137) = 148

iii. g(153) = 162

iv. g(271) = 281

v. g(933) = 948

8.2.2 Here is the list, obtained with the help of a computer:

1, 3, 5, 7, 9, 20, 31,42, 53,64, 75, 86.97.

Note the suggestive pattern!

9.2.1 The orbits are:

i. 3,13,1113,3113,2123,112213,312213,212223,114213,31121314,41122314,3

   We have reached the fixed point 21322314.

ii. 4,14,1114,3114,211314,31121314,41122314,31221324,21322314,21322314

   We have reached the same fixed point!

iii. 9,19,1119,3119,211319,31121319,41122319,3122131419,4122231419,3132

   We have reached a different fixed point this time—the self-referential number 3122331419.

9.2.2 A fixed point is reached in each case:

i. 20,1012,102112,103122,10212213,10313213,10311233,10311233,....

ii. 25,1215,211215,312215,21221315,31321315,31123315,31123315,....

iii. 95,1519,211519,31121519,4112131519,4112131519,....

9.2.3 Cycles are reached with these seeds:

i. 40,1014,102114,10311214,1041121314,1051121324,104122131415,10512

We have found a 2-cycle, namely:

⟨104122232415,103142132415⟩.

ii. 50,1015,102115,10311215,1041121315,105112131415,106112131425,1051

This time we have reached a 3-cycle!

⟨10414213142516,10412213341516,10512223241516⟩.

9.2.4 ***A MATHEMATICA*** program to do the calculations is given below.

```
selfdescribe [a-List :] :=
    (b = Sort [a];c = Union[b,{}];
    d = Length [c];
    f[i-];= Count [b,c[[i]]];g = {};
    Do [g = Join [g,{f[i],c[[i]]}], {i,1,d} ],g )
    final [a-List,n-] := NestList [selfdescribe, a,n]
```

Using the program, we find that the seed 160 gives rise to the following 2-cycle:

⟨1051421314152617,1061221324251617⟩.

The seed 129 also gives rise to a 2-cycle:

⟨314213241519,412223241519⟩.

The seed 314159 gives rise to a 3-cycle:

⟨51222314251619,41421314251619,51221334151619⟩.

Clearly a lot more such data can be produced by further experimentation.

9.2.5 Please refer to the solution to Problem 14.2.2.

10.1.1 If h(x) = x, then - x = x, therefore x = 0. This means that ⟨-1⟩ is the only l-cycle.

10.1.2 Any function of the type h(x) = a - x, where a is any number, is a 2-cycle function.

10.1.3 Here is one such function: let k(x) be defined so that

$$k(x) = x + 1 \text{if}[x]\text{isodd}, x - 1 \text{if } [x]\text{is even}.$$

Then k(x) is a 2-cycle function, but it has no l-cycle. Any odd integer may be substituted for the '1'.

10.1.4 Iterating the function, we obtain:

$$x \mapsto 1\ -1\ x = x - 1\ X \mapsto 1 - x\ x - 1 = 1\ 1 - x \mapsto 1\ -1 - x\ 1 = x,$$

and we see that the cycle has closed. Thus every number is caught in a 3-cycle. unless it happens to be a fixed point. Are there any fixed points? To check, we examine the equation $x = (x - 1)/x$, which simplifies to $x2 - x + 1 = 0$. However this equation has no solutions in real numbers. (Proof: $x2 - x + 1 = (x - 1/2)2 + 3/4$, so $- x2 - x + 1 \geq 3/4$ for all real values of x. In particular we can never have $x2 - x + 1 = 0$.) We conclude that there are no fixed points, so each number is part of a 3-cycle. Thus the given function is a 3-cycle function.

To see why we need to have $x \neq 0,1$, simply examine the denominators of the expressions listed above. No zeroes can appear here. so the restrictions are needed just as listed.

10.1.5 The function m(x) is a 2-cycle function:

$$x \mapsto x\ x - 1 \mapsto x/(x - 1)\ x/(x - 1) - 1 = x.$$

10.1.6 The orbit in this case is:

$$x \mapsto 1\ 1 - x \mapsto 1\ 1 - 1/(1 - x) = x - 1\ x \mapsto 1\ 1 - (x - 1)/x = x.$$

It follows that n(x) is a 3-cycle function.

10.2.1 $p(3.1) = 2.1, p(-7.3) = -6.3, p(0) = 1$.

10.2.2 Let $x = k + z$, where $k = [x]$ is the integral part of x and z is the fractional part of x; thus. $0 \leq z < 1$.

If k is odd, then $p(x) = k - 1 + z$. The integral part of $k - 1 + z$ is $k - 1$. which is even, so $p(k - 1 + z) = (k - 1 + z) + 1 = k + z = x$, that is, $p(p(x)) = x$. The argument for the case when k is even is practically the same and we do not give it here. Since [x] and [p(x)] have opposite parity (that is, if one of them is odd then the other is even). it is not possible for p(x) and x to be equal. Thus $p(x) \neq x$ for all values of x and p has no 1-cycles.

10.2.3 p(x) = 3.3 holds only when x = 2.3.

10.2.4 Let x = k + z, where k = [x] is the integer pan of x, and let k leave remainder r when divided by 5. We compute the value of p(x) by the following rule:

$$p(x) = x + 1 \text{ifr} = 0,1,2,3;x - 4 \text{if } r = 4.$$

Here is a typical orbit: $10.7 \mapsto 11.7 \mapsto 12.7 \mapsto 13.7 \mapsto 14.7 \mapsto 10.7 \mapsto \cdots$.

It is not too hard to see why p is a 5-cycle function. Please fill in the details on your own.

10.2.5 For an input value of x, let k = [x] be the integral part of x, and let r be the remainder when k is divided by n. We compute the value of f(x) as shown below:

$$f(x) = x + 1 \text{ ifr} = 0,1,2,\ldots,n - 2;x - n + 1 \text{ if } r = n - 1.$$

We leave as an exercise the verification that f is an n-cycle function.

10.3.1 This is clear , because $1/(1/x) = x, 1 - (1 - x) = x, -(-x) = x$.

10.3.2 Possible square roots are listed alongside the functions.

i. a(x) = x + 1; possible square root: b(x) = x + 1/2.

ii. a(x) = x + 2; possible square root: b(x) = x + 1.

iii. a(x) = 4x; possible square root: b(x) = 2x.

iv. a(x) = 2x; possible square root: b(x) = x2.

10.3.3 Given a function a, a function b such that b ◦ b ◦ b = a (also written as bbb = a or b(3) = a or b(b(b(x))) = a(x) for all values of x) can be referred to as a *cube root* of a.

For the function f, where f(x) = -x,f is itself a cube root! (Thus the function is its own cube root.) Other cube roots can be manufactured using the same technique as was used for the square root of the same function (as shown earlier in the text).

10.5.1 This will become clear when we examine the orbit of x. We give the argument only for positive values of x.

Let n be the unique integer such that 10n ≤ x < 10n+1. Suppose that n is even; then x↦10x. Next, for the number 10x the corresponding number n′ is n + 1, which is odd since n is even. Therefore 10x↦ - (10x⁄10) = -x, and so b has taken x to - x in two steps in this case. Continuing, we find that b takes - x to x in another two steps.

We argue in a similar manner to cover the other possibilities.

10.5.2

i. a(x) = x2 (domain: all real numbers).

   One possibility is the function g(x) = x2. With this choice,

   g(g(x)) = x2 2 = x2,

   that is, g ◦ g = a.

ii. a(x) = x⁄(2x + 1) (domain: all numbers x≠ - 1⁄2).

   One possibility is the function g(x) = x⁄(x + 1). This leads to:

   x↦ x x + 1↦ x⁄(x + 1) x⁄(x + 1) + 1 = x 2x + 1,

   and so g ◦ g is the given function a. Note that we must exclude both - 1 and - 1⁄2 from the domain of g.

iii. a(x) = 1⁄x (domain: the set of non-zero real numbers).

   Let f be any function from **R** into R such that f(f(x)) = -x for all x. We have already indicated in the text how to construct one such function f, and we shall make use of this function now. Define the function a by

   g(x) = 10f(log 10x) ifx > 0, - 10f(log10(-x)) ifx > 0,

   (Note that g(0) must of necessity remain undefined; this is because 1⁄x is not defined for x = 0.)

   We claim that g(g(x)) = 1⁄x for all x≠0. This is easy to prove. Assume that x > 0. Then g(x) = 10f(log10x), so log10g(x) = f(log10x). It follows that

   g(g(x)) = 10f(log 10g(x))

   = 10f(f(log 10x)

$$= 10- \log 10x = 1\ x,$$

by the definition of the logarithm function. The case when $x < 0$ is handled similarly, and we omit the details. It follows that $g \circ g = a$.

10.5.3 Define f (0) to be 0; for n > 0 let

$$f(n) = n + 1 \text{ if } n \not\equiv 0 (\text{mod} 4),\ - n + 3 \text{ if } n \equiv 0 (\text{mod } 4);$$

and for n < 0 let f(n) = -f(-n). Here is a picture of how f acts:

$$1 \to 2 \to 3 \to 4 \to -1 \to -2 \to -3 \to -4 \to 1,$$

$$5 \to 6 \to 7 \to 8 \to -5 \to -6 \to -7 \to -8 \to 5,....$$

The proof that f(4)(n) = -n for all integers n is left to the reader.

10.5.4 This problem is solved in Chapter 18.

10.5.5 This problem is solved in Chapter 17.

11.1.1 We have listed below the iteration sequences resulting from the seed value x = 1.

i. Function: $f(x) = 1 + x/2$; orbit: ⟨1,1.5,1.75,1.875,1.938,1.969,1.984,1.992,1.996,1.998,1.999,...⟩.

   The limit of the sequence is 2.

ii. $f(x) = 1 + x$; orbit: ⟨1,1.4142,1.5538,1.5981,1.6118,1.6161,1.6174,1.6179,1.618,1.618,...⟩.

   The limit of this sequence is approximately 1.618034…, as may be seen by extending the calculations further. This is the so-called *golden ratio*, and it can be expressed in the nicer (and more illuminating) form $(5 + 1)/2$.

iii. $f(x) = 1 + 2x$; orbit: ⟨1,1.732,2.113,2.286.2.361,2.392,2.405,2.41,2.413,2.4136,2.4139,…⟩.

   The limit of the sequence is approximately 2.41421536…, and it can also be expressed as 2 + 1.

iv. $f(x) = 1 + 1/x2$; orbit:

⟨1,2,1.25,1.64,1.372,1.531,1.426,1.492,1.449,1.476,1.459,1.47,...,⟩.

The limit of the sequence is approximately 1.46557…, and it can also be expressed as

$$1\ 3 + 29 + 393\ 54\ 1/3 + 29 - 393\ 54\ 1/3\ ,$$

only, this form may not be particularly illuminating!

v. f(x) = 1000 + x3; orbit: ⟨1,10.0033,10.0332,10.03333,…⟩.

The limit of the sequence is approximately 10.03333… and it can be expressed as

$$500 + 20249997\ 9\ 1/3 + 500\ -20249997\ 9\ 1/3$$

but, as earlier this may not be not too illuminating!

11.1.2

i. f(x) = 1 + x/2.

If there is a limit at all, say, L, then it must satisfy the equation L = 1 + L/2, whose solution is L = 2. (We had already observed that the limit appears to be 2.) To show that the sequence of iterates converges to 2, note that from the defining relation f(x) = 1 + x/2, we obtain

$$f(x) - 2 = x\ 2 - 1 = 1\ 2(x - 2).$$

This shows that the gap between f(x) and 2 is 1/2 the gap between x and 2. It follows that the gap diminishes to 0 as the iteration progresses, and so the limit is 2, as stated.

ii. f(x) = 1 + x.

If there is a limit at all, say L, then it must satisfy the equation L = 1 + L or L2 - L - 1 = 0. This has two solutions, namely:

$$L = 1\ \pm 5\ 2\ .$$

Since we start with a positive seed (x = 1) all the iterates are positive and so convergence will be, if at all, to (1 + 5)/2 and not to the negative root. Write, for convenience, α = (1 + 5)/2. We first claim that if

$$0 < x < \alpha \text{ then } x < 1 + x < \alpha.$$

This is not too hard to prove and we leave the details to the reader. (Hint: Examine the graphs of $y = (1 + 5)/2$ and $y = x$.) Next, we show that

$$\text{if } 0 < x < \alpha \text{ then } \alpha\ -1 + x\ \alpha - x < \alpha - 1\ \alpha\ .$$

The hint given above serves to show the way out here too: the graph of $y = 1 + x$ is concave (arching upwards). Let A,P and B be the points (0,1), (x,1 + x) and (α,α) respectively; here $0 < x < \alpha$, so P lies in-between A and B on the curve. Since the curve is concave, the slope of PB is less than that of AB. This immediately leads to the inequality stated above.

Now $\alpha \approx 1.618 < 2$, so $(\alpha - 1)/\alpha < 1/2$. It follows that

$$\text{if } 0 < x < \alpha \text{ then } \alpha\ -1 + x\ \alpha - x < 1\ 2.$$

Thus the gap between α and f(x) is *less* than half the gap between α and x. It follows that the iterates converge to α, as claimed.

iii. $f(x) = 1 + 2x$.

If there is a limit at all, say L, then it must satisfy the equation $L = 1 + 2L$ or $L2 - 2L - 1 = 0$, whose solution is $L = 1 \pm 2$. As in (ii), we argue that convergence, if it takes place at all, must be to $1 + 2$. From this point on, the argument proceeds just as in (ii). Write for convenience $\beta = 1 + 2$ and observe that if $0 < x < \beta$ then $x < 1 + x < \beta$. Next, note that the curve $y = 1 + 2x$ is concave, so if A,P and B denote the points (0,1),(x,1 + x) and (β,β) respectively, where $0 < x < \beta$, then the slope of PB is less than the slope of AB. It follows that

$$\text{if } 0 < x < \beta \text{ then } \beta\ -1 + x\ \beta - x < \beta - 1\ \beta\ .$$

Now

$$\beta\ \beta - 1 = 2\ 2 + 1 = 2\ -2 \approx 0.59 < 0.6,$$

so (referring to the gap between β and x as the 'error'), the error shrinks at each iteration to 3/5 or less of its old value. This shows that the error diminishes to 0 as the iteration progresses, and so the iterates converge to $1 + 2$ as claimed.

iv. $f(x) = 1 + 1/x2$.

If there is a limit at all, say L, then it must satisfy the equation L = 1 + 1⁄L2 or L3 - L2 - 1 = 0. The real solution of this equation is

$$L = 1\ 3 + 29 + 393\ 54\ 1/3 + 29 - 393\ 54\ 1/3 \approx 1.46557.$$

Let α denote this number. The reader will have noted that the orbit ⟨1; f ⟩ is an *oscillatory* one: it does not go steadily in one direction but oscillates about α. To show convergence to α we shall have to prove some inequality for the error term |x - α|, just as we did for (ii) and (iii), above. This tums out to be rather more complicated that for the earlier cases.

We observe firstly that if $1 \le x < \alpha$ then $\alpha < f(x) \le 2$, while if $x > \alpha$ then $1 < f(x) < \alpha$. Since we start with a seed of 1, this shows that each iterate lies between 1 and 2. Now if $1 \le x < 2$ then $1.25 < f(x) \le 2$. This in turn implies that $1.25 \le f(f(x)) < 1.64$ and, continuing, $1.37 < f(f(f(f(x)))) < 1.64$. Thus from the third iterate onwards, each x-value lies between 1.37 and 1.64. Now the graph of y = 1 + 1⁄x2 is *convex*, that is, arching downwards, so in the interval $1.37 \le x \le 1.64$, the point where the curve is steepest (in absolute value, ignoring the sign of the slope) must be one of the two end-points; indeed, the point is clearly x = 1.37. To compute the slope of the curve at this point we use calculus. (Its use can be avoided, but this makes matters more complicated.) With f(x) = 1 + 1⁄x2, we obtain $f'(x) = -2/x3$ and so $f'(1.37) = -2/1.373 \approx -0.78$; thus $|f'(1.37)| < 0.8$. It follows that for any x with $1.37 \le x \le 1.64$,

$$\alpha - f(x)\ \alpha - x < 0.8.$$

Thus the error |x - α| shrinks to 4/5 or less of its old value at each stage. It follows that the error diminishes to 0, and so the iterates tend to α, as claimed.

v. f(x) = 1000 + x3.

If there is a limit at all, say L, then it must satisfy the equation L3 = 1000 + L, whose sole real solution is

$$\alpha = 500 + 20249997\ 9\ 1/3 + 500\ -20249997\ 9\ 1/3$$

(α ≈ 10.03333). To show that the iterates converge to α, we follow the strategy of (ii), (iii), above. This works because the curve y = 1000 + x3 slopes upwards and is concave, just like the curves y = 1 + x and y = 1 +

2x. The details are omitted, as the style of proof is substantially the same.

For more on this type of convergence proof, the reader should study Chapter 16.

11.1.3 The orbit of the pair (x,y) is as shown below:

$$(x,y) \mapsto y, y+1-x$$
$$\mapsto y+1-x, x+y+1-xy$$
$$\mapsto x+y+1-xy, x+1-y$$
$$\mapsto x+1-y, x$$
$$\mapsto (x,y)$$

and the cycle has closed. (The factorization $xy + x + y + 1 = (x + 1)(y + 1)$ turns out to be useful at one point.) A sample orbit, using numbers instead of symbols:

$$(1,2)\mapsto(2,3)\mapsto(3,2)\mapsto(2,1)\mapsto(1,1)\mapsto(1,2).$$

11.1.4 Clearly, what we need to avoid is division by zero. This means that none of the quantities $x, x + 1, y, y + 1, x + y + 1$ should.be zero. Therefore the following restrictions are needed:

$$x\neq 0,-1, y\neq 0,-1, x+y+1\neq 0.$$

11.2.1 Here is a program in *BASIC* that will help you do this problem.

```
REM   collatz
CLS
FOR n = 2 TO 1000
    X = n:  count = 0
    WHILE x > 1
        y = x MOD 2
        IF y = O THEN x = x / 2
         IF Y = 1 THEN x = 3 * x + 1
```

```
        count = count + 1
    WEND
    PRINT "("; n; ", "; count; "), "
NEXT n
```

On running the program, we find that the largest number of steps needed to reach 1 occurs for the seed 871 (178 steps). For seeds in the range 1… 100, the maximum number of steps occurs for the seed 97 (118 steps).

11.3.1 By the definition of cycle, if we apply f to a set C of numbers that form a cycle, we get back the set C itself. Thus if D is the union of all such sets, then. f(D) = D. This means that if we start with a large set S, large enough to contain all of D, then we can be certain that f(S) too will contain all of D. Further, any set E for which the relation f(E) = E holds must consist entirely of numbers that form part of some cycle.

Now consider the sequence S,S1 = f(S),S2 = f(S1),…. The set of images obtained when a function is applied to a finite set can never be larger than the set itself, so the cardinalities of the Si form a non-increasing sequence:

$$|S| \geq |S1| \geq |S2| \geq \cdots.$$

On the other hand the set D is contained in each Si, so $|Sj| \geq |D|$ for all i. So we have a sequence of positive integers whose initial term is the size of S, which never increases, and whose terms are all $\geq |D|$. At some stage, therefore, two successive terms must become equal, which means that we will have at hand a set T for which f(T) = T. As per the comment made earlier, this set comprises all the elements that form part of some cycle. The cycles can now be unraveled from T in a straightforward manner.

11.3.2 On running through the computations, we obtain the following. Let S be the set {0,1,2,3,…,99}, and let S1 = g(S),S2 = g(S1),S3 = g(S2),…. Then:

S1 = {0,2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,34,36,40,42,44,48,50,52,54,56,60,66,

S2 = {0,2,4,6,8,10,12,14,16,18,20,24,30,36,40,42,50,52,56,66,70,72,90};

S3 = {0,2,4,6,12,14,18,20,24,30,40,42,50,56,70,72,90};

S4 = {0,4,6,12,14,18,20,30,42,50,56,70,90};

S5 = {0,4,6,12,14,18,20,30,42,56,70,90};

S6 = {0,4,6,12,14,18,20,42,56,70,90};

S7 = {0,4,6,12,14,18,20,42,56,70,90} = S6.

Thus the set of numbers that belong to some cycle is the set

$$T = \{0,4,6,12,14,18,20,42,56,70,90\}.$$

Applying g to these numbers gives the same numbers but in the following order:

$$0,20,42,4,18,70,6,14,12,56,90.$$

It follows that g has two 1-cycles, $\langle 0 \rangle$ and $\langle 90 \rangle$, and one 9-cycle:

$$\langle 4,20,6,42,14,18,70,56,12 \rangle.$$

11.4.1 Clearly D(n) = 1 if and only if n has no proper divisors other than 1, which means that n = 1 or a prime. The required numbers are therefore 1, 2, 3, 5, 7, 11, 13, ….

11.4.2 Using the prime factorizations of the numbers ($10 = 2 \times 5$,$20 = 22 \times 5$,$30 = 2 \times 3 \times 5$,$40 = 23 \times 5$,$50 = 2 \times 52$), we obtain:

$$D(10) = (1 + 2) \times (1 + 5) - 10 = 8,$$

$$D(20) = (1 + 2 + 4) \times (1 + 5) - 20 = 22,$$

$$D(30) = (1 + 2) \times (1 + 3) \times (1 + 5) - 30 = 42,$$

$$D(40) = (1 + 2 + 4 + 8) \times (1 + 5) - 40 = 50,$$

$$D(50) = (1 + 2) \times (1 + 5 + 25) - 50 = 43.$$

It is easy to check that the stated values are correct.

11.4.3 The rule works because when we take the product $P \times Q \times R \times \cdots$, we obtain all possible divisors of n, including n itself, which therefore must be subtracted out. For instance, consider n = 50; multiplying out the expression $(1 + 2) \times (1 + 5 + 25)$ we obtain $1 + 2 + 5 + 10 + 25 + 50$, and the six summands are precisely the six divisors of 50.

11.4.4 Suppose that n is odd. Then 2n has the prime factor 2, so

$$D(2n) = (1 + 2) \times (D(n) + n) - 2n = 3D(n) - n,$$

and it follows that if $D(n) > n$, then $D(2n) > 2n$; the stated inequality thus holds.

Next, suppose that n is even, say $n = 2a\text{-}1pbqc\ldots$, where p,q,… are odd primes and $a > 1$. Then $2n = 2apbqc\ldots$. Using the identity $1 + 2 + 22 + \cdots + 2k\text{-}1 = 2k - 1$ and writing for convenience

$$X = (1 + p + p2 + \cdots + pb) \times (1 + q + q2 + \cdots + qc) \times\cdots,$$

we obtain

$$D(n) = X(2a - 1) - n,$$

$$D(2n) = X(2a+1 - 1) - 2n.$$

Now $2a+1 - 1 > 2(2a - 1)$, so $D(2n) > 2(D(n) + n) - 2n$, that is, $D(2n) > 2D(n)$. Since $D(n) > n$ it follows that $D(2n) > 2n$.

Since 12 is abundant (because $1 + 2 + 3 + 4 + 6 > 12$) , it follows that each member of the sequence 24, 48, 96, 192, 384, … is abundant. Thus there are infinitely many abundant numbers.

Note that the number of deficient numbers too is infinite, because $D(p) = 1 <$ p for every prime p, and (of course) there are infinitely many primes.

11.4.5 The equation $D(n) = 2$ can never hold.

11.4.6 With P of the stated form and $n = P(P + 1)/2$, we have $N = 2k\text{-}1(2k - 1)$ and the rule gives

$$D(N) = 1 + 2 + 22 + \cdots + 2k\text{-}1 \times 1 + 2k - 1 - N,$$

that is, $D(N) = (2k - 1)2k - (2k - 1)2k\text{-}1 = (2k - 1)2k\text{-}1 = N$. Thus N is perfect.

11.4.7 We are told that N is even, so we write $N = 2aB$ where B is odd and $a > 0$. Using the rule described in the text, we obtain

$$D(N) = (2a+1 - 1)(D(B) + B) - 2aB.$$

Since N is perfect, $D(N) = N = 2aB$, so

$$(2a+1 - 1)(D(B) + B) = 2a+1B.$$

Therefore we have

$$D(B) + B\,B = 2a+1\ 2a+1 - 1, D(B)\,B = 1\ 2a+1 - 1.$$

The fraction on the right has no cancellable terms, so we must have

$$D(B) = c, B = c \cdot (2a+1-1)$$

for some integer $c > 0$. If $c > 1$, then since c is a proper factor of B, we must have $D(B) \geq c + 1 > c$, a contradiction. Therefore $c = 1$. It follows that $B = 2a+1 - 1$ and that $D(B) = 1$,which means that B is prime. Thus N takes the form $P(P + 1)/2$ where $P = 2a+1 - 1$ is a Mersenne prime. This is exactly what Euler had asserted.

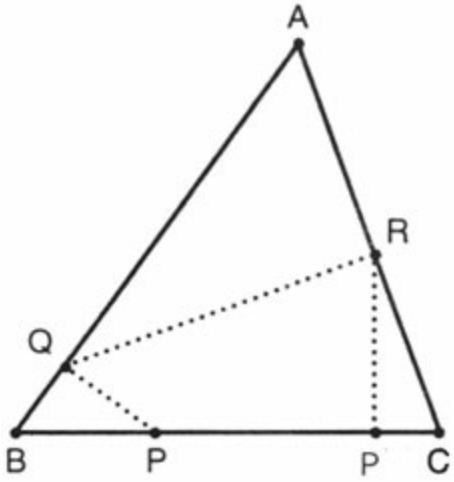


Figure B.1. *Iteration with perpendiculars*

11.4.8 We have, $6232 = 23 \cdot 19 \cdot 41$ and $6368 = 25 \cdot 199$, so

$$D(6232) = (15 \cdot 20 \cdot 42) - 6232 = 6368,$$

$$D(6368) = (63 \cdot 200) - 6368 = 6232.$$

11.4.9 We have, $9363584 = 27 \cdot 191 \cdot 383$ and $9437056 = 27 \cdot 73727$, so

$$D(9363584) = (255 \cdot 192 \cdot 384) - 9363584 = 9437056,$$

$$D(9437056) = (255 \cdot 73728) - 9437056 = 9363584.$$

11.4.10 This has already been done in the text.

13.2.1 The sides of the inscribed triangle are (respectively) at right angles to the sides of $\triangle ABC$ and therefore enclose the same angles. It follows that the

two triangles are similar.

13.2.2 If $\triangle ABC$ is right-angled, we reach the limiting shape at the very first stage.

13.2.3 If $\triangle ABC$ is obtuse-angled, then the limiting shape juts outside it.

13.2.4 Empirically we find that exactly the same thing happens: we reach a limiting shape, which in this case is a quadrilateral with one vertex on each side of the original quadrilateral.

13.2.5 We consider the iteration where the movement is in a clockwise direction—i.e., we drop perpendiculars in turn to AB, to CA, to BC,… (see Figure B. 1). We now have:

$$BP = x, \therefore BQ = x\cos B,$$

$$\therefore AQ = c - x\cos B, \therefore AR = (c - x\cos B)\cos A,$$

$$\therefore CR = b - (c - x\cos B)\cos A,$$

$$\therefore CP' = (b - (c - x\cos B)\cos A)\cos C,$$

$$\therefore BP' = a - (b - (c - x\cos B)\cos A)\cos C,$$

$$\therefore BP' = a - b\cos C + c\cos A\cos C - x\cos A\cos B\cos C.$$

Since $a - b\cos C = c\cos B$, the last equation simplifies to

$$BP' = c(\cos B + \cos A\cos C) - x\cos A\cos B\cos C.$$

So the iteration follows the rule given below:

$$x \mapsto c(\cos B + \cos A\cos C) - x\cos A\cos B\cos C.$$

Write $\alpha = \cos A, \beta = \cos B, \gamma = \cos C$; the mapping is then:

$$x \mapsto c(\beta + \alpha\gamma) - (\alpha\beta\gamma) \cdot x.$$

Since $|\alpha\beta\gamma| < 1$ (true because α, β, γ are cosines of angles), it follows that the iteration converges, with $x \to x0$ where

$$x0 = c\ \beta + \alpha\gamma\ \alpha\beta\gamma\ .$$

We conclude that the sequence $\langle P, f(P), f(f(P))\ldots\rangle$ converges to a point P^* with $BP^* = x0$.

13.3.1 Since the triangles Ti tend to the equilateral shape, their incenters Ij converge to the center of the circumcircle of △ABC.

13.3.2 From the relation f(x) = 90 - x⁄2 we deduce that

$$f(x) - 60 = 30 - x/2 = 60 - x\ 2\ ,$$

that is, |f(x) - 60| = |x - 60|⁄2. So the gap between x and 60 gets halved after each iteration, and the x-values converge to 60.

13.4.1 As noted in the text, the triangles Ti tend to the equilateral shape, therefore their centroids Gi converge to the center of the circumcircle of △ABC. However convergence is hard to prove.

13.4.2 The same result holds as in 13.4.1, but as earlier convergence is hard to establish.

13.5.1 (Please draw a figure for yourself!) Now label T0 as △ABC,T1 as △A1B1C1,T2 as △A2B2C2 and T3 as △A3B3C3. The proof that T3 ~T0 works out very simply after joining AP, for it turns out that ∠PAB = ∠PA3B3 and ∠PAC = ∠P443C3. (To see why, examine the many cyclic quadrilaterals concealed in the figure: PA1BC1,PA2B1C2,PA3B2Cγ,…. Now make repeated use of the theorem which states that angles in the same segment of a circle are equal.) Therefore ∠BAC = ∠B3A3C3, and likewise for the other two angles. Thus T0 ~T3 as claimed.

The result can be extended. For a convex n-sided polygon P0 and a fixed interior point P, construct P1 by joining (in sequence) the feet of the perpendiculars from P to the sides of P0. Likewise construct P2 by dropping perpendiculars from P to the sides of P1; and so on, … Then Pn is similar to P0.

13.6.1 The classes are: (i) acute-angled triangles (ii) right-angled triangles (iii) obtuse-angled triangles. In (ii) the orthocenter invariably lies at the vertex corresponding to the right angle; the orthocenter can never be situated in the interior of a side.

13.6.2 The Ti's do not converge to a limiting shape in this iteration.

13.6.3 Through each vertex of the given △ABC, draw a line parallel to the side opposite it. The three lines enclose a triangle, say △XY Z. This triangle is

homothetic to △ABC and A,B,C lie at the mid-points of its sides. The altitudes. of △ABC are now seen to be the perpendicular bisectors of the sides of △XY Z and therefore concur at the circumcenter of △XY Z.

13.6.4 We first prove Euler's theorem: for any triangle ABC the orthocenter (H), the centroid (G) and circumcenter (O) are collinear with OH = 3OG.

Let the mid-points of BC,CA, *AB* be A′,B′,C′; then O is the orthocenter of △A′B′C′; the triangles ABC,A′B′C′ share the same centroid, namely, G; and AG/GA′ = BG/GB′ = CG/GC′ = 2.

Consider the homothety with center G and scale factor - 1/2 (that is, the image of any point X under the homothety is the point X′ with GX′ = XG/2 and with G lying on segment XX′; in vector terminology, **GX′** = -**GX**/2 〉 . The homothety maps △ABC to △A′B′C′ and the orthocenter of △ABC to that of △A′B′C′. That is, it maps H to O, and it follows from this that **GO** = -**GH**/2. This proves Euler's theorem.

We also prove the following rather trivial result: *let* G *be an arbitrary point lying within the interior of* Γ. *Then there exists a triangle within* Γ *whose centroid is* G.

Proof: Let O be the center of Γ and let the ray OG meet Γ at A. Extend AG to A′ where GA′ = AG/2, and draw a line through A′ perpendicular to AA′. Let this line meet Γ at B,C. Then △ABC has centroid G.

These two results imply the statement to be proved. We leave the details to the reader.

13.7.1 If *ABCD* is a square and P,Q,R,S are located on the sides AB,BC,CD,DA so that AP/PB = BQ/QC = CR/RD = DS/SA = k, then △SAP is congruent to △PBQ and SP = PQ,SP ⊥ PQ. This is enough to conclude that PQRS is a square.

13.7.2 A sequence of 'nested diamonds' results.

13.7.3 This, naturally, will depend upon the, eye of the beholder, but to the author the nicest pictures are obtained with k close to 0 or 1, but not *too* close; k = 0.1 or k = 0.9 seem just right. (With k = 0.01 or smaller the picture is too dense, and with k = 0.3 or so there is too much 'white space'.)

13.7.4 The same results hold: convergence to a point. (The successive quadrilaterals are not necessarily similar to one another.)

13.7.5 A curve can be thought of in two different ways: *as made up of its points*, or as *enveloped by its tangent*. The curve described in the problem is an example of a curve being enveloped by tangents. We list a few more illustrative examples.

- The *caustic curve* seen in a tea-cup or a vessel half filled with milk when a light-source is held near it (the curve is seen on the surface of the liquid) provides a particularly nice example of a curve enveloped by its tangents. It is well known that the curve is a cusped one.
- Another fine example is provided by the drawing often made by children while doodling in their notebooks: let ℓ,ℓ' be lines meeting at a point O; let n be a positive integer (say n = 20 or so) and let h be a small positive number; let A1,A2,A3,…, An be distinct points on ℓ and let B1,B2,B3, …,Bn be distinct points on ℓ', with

  $$OA1 = A1A2 = \cdots = OB1 = B1B2 = \cdots = h.$$

  Let the lines A1Bn,A2Bn-1,A3Bn-2,…,AiBn-1-i,… be drawn. The various lines will be seen to very gracefully envelope a curve; the larger the value of n and the smaller the value of h, the nicer the effect. (In fact the curve is a parabola, but this needs proof.)
- Another such curve is seen in an artifact frequently made in the Crafts department of a school, in its embroidery section. Let C be a circle, let n be a positive integer (say n = 20) and let points A1,A2,A3,…,An be marked symmetrically on C, with A1A2 = A2A3 = ⋯ = An-1An = AnA1. Let line segments be drawn according to the following scheme: we join A1A2,A2A4,A3A6,A4A8,…,AjA2i,…. The various lines will be seen to envelope a cusped curve. Modifications are easy to make, by choosing different pairs of points to join. (In the embroidery section, the lines are threads of different colours, and the points are nails around which the threads are looped. The overall effect is a very pretty one.)
- A *rainbow* is an excellent example of a caustic, but we leave the details to the reader.

13.7.6 The connection of this problem with the iteration being studied should be clear: at each instant, the positions occupied by the four men are the vertices of a square, and the square progressively narrows down to the center of the square. Each man traces out a spiral Which proceeds from a vertex of the square towards its center. The spirals belong to a family called *equiangular spirals*. A nice property of the configuration is that the length of the portion of the curve between a vertex of the square and the center is the same as the side of the square.

13.7.8 There is no convergence towards 'equilateral-ness'. (This is easy to check empirically.)

13.9.1 If $A'B' \parallel AB$ and $B'C' \parallel BC$, then the lines AA', BB', CC' and DD' concur at a point, say O; this is the desired fixed point.

If $A'B' \perp AB$ and $B'C' \perp BC$, then we draw two circles, one with AA' as diameter, the other with BB' as diameter. These circles will intersect in two points, of which only one, say O, will lie inside M'; this is the desired fixed point.

13.9.2 The fixed point does exist, but it may not lie inside the given map, M. To see why, make the original map part of a larger map N, large enough so that *its* image, N' , does lie inside N. A fixed point will now certainly exist, but it may not lie inside M . (That is, it may lie in the region inside *N* but outside M.)

13.9.3 Let the positive x-axis represent East, and the positive y-axis North. Call the origin *O*, and the terminal point $T(x,y)$; let A be the point (1, 0) , reached after the first step (1 km East). Then AT must have *half* the length of OT and be rotated 90° relative to it. Since $\overrightarrow{OT} = (x,y)$ and $\overrightarrow{AT} = (x - 1,y)$, we get

$$(x - 1,y) = 1\ 2 \cdot (-y,x), \therefore x - 1 = -y\ 2, y = x\ 2, \therefore x = 4\ 5, y = 2\ 5.$$

So the terminal position is T(4⁄5,2⁄5).

14.2.1 The analysis is done in much the same manner as was done in Problem 11.3.1, and we leave the details to the reader.

14.2.2 We start by analyzing a slightly different problem, in which we deal with *lists* of numbers of a fixed length (tuples") of a fixed length), say m + 1.

Specifically, we deal with the following iteration which we call SR. For any (m + 1)-tuple of numbers, say **a**, let SR (a) be computed thus: for each i in the set {0,1,2,…,m} we count the number of i's in a, then add k to this number (k being some fixed number). Let the resulting number be c(i).Then the tuple SR(a) is given by

$$SR(a) = (c(0),c(1),c(2),\ldots,c(m)).$$

Note that SR(a) is a tuple of the same length as **a**. This invites us to compute the orbit ⟨a; SR ⟩ and to study the properties of this iteration.

Example 1 . With m = 3,k = 1 and **a** = (1,0,1,2) , we obtain SR(a) = (2,3,2,1). Further iterations give:

$$(2,3,2,1)\mapsto(1,2,3,2)\mapsto(1,2,3,2)\mapsto\cdots.$$

Note that a fixed point has been reached.

The ***MATHEMATICA*** commands we have used to do the computations are displayed below.

```
selfrefer[a_List,m_,k_] :=
        (f[i_] := Count [a,i] ;b = {};
        Do [b = Join [b,{f[i]}],{i,0,m} ] ;
        c =b+k;c )
SR ⌊a-LiSt⌋ := selfrefer [a,    6,    1]
```

Here is the result of iterating SR on (1, 1. 1, 1. 1, 1.2) (the ***MATHEMATICA*** command is:

Nest List ⌈SR,a,{1,1,1.1,1,1.1,2},10 ⌉;

the "10" indicates the number of iterations):

(1,1,1,1,1,1,2),(1,7,2,1,1,1,1),(1,6,2,1,1,1,1),

(1,6,2,1,1,1,2),(1,5,3,1,1,1,2),(1,5,2,2,1,2,1),

(1,4,4,1,1,2,1),(1,5,2,1,3,1,1),(1,5,2,2,1,2,1),

(1,4,4,1,1,2,1),(1,5,2,1,3,1,1),(1,5,2,2,1,2,1),….

Note that a 3-cycle has been reached.

Empirically we find that we reach a fixed point or a cycle for any starting tuple whatever, and we now show why this must always happen.

For any tuple a, the following is rule: the number of i's in a is one of the numbers 0,1,2, …, m + 1, therefore c(i) is one of the numbers k,k + 1,k + 2, …,k + m + 1. Thus for each i,c(i) assumes only one of a list of m + 2 values. The total number of (m + 1)-tuples in which each entry is one of these m + 2 numbers is clearly (m + 2)m+1, which though large is a *finite* number. Let *S* denote the set of tuples (a0,a1,a2,…,am) where for each i,ai is one of the numbers k,k + 1,k + 2,…,k + m + 1. Then SR (a) ∈ S for any (m + 1)-tuple a and SR is amapping of S into itself. It follows that S plays the role of the 'black-hole' set referred to earlier in the text. Following the same logic used, we infer that we shall inevitably reach a cycle in all cases. ❑

We shall leave the reader with the task of making the connection between this iteration and the one presented earlier in the text. The alert reader will notice that the present iteration is a far better behaved one: the definition is precise, and there is no possibility of ambiguity. In the earlier case, there would be scope for ambiguity if for some i the number of i's in the current number were to exceed 9, for then the digital representation of the iterate would be unclear. In the iteration presented here we would not be faced with ambiguity even if such a thing happened, as the iteration is not a digit-based one.

14.2.3 The logic used for this problem is once again be of the type used in Problem 11.3.1.

For x = 1,2,3,…,20,21 let f(x) denote the position number of the card that was originally at position number x, after one iteration of the procedure. It is easy to check that f assigns values as per the following table. (For instance, a card originally in position 1 will move to position 8, and so on.)

x	1	2	3	4	5	6	7	8	9	10	11
$f(x)$	8	8	8	9	9	9	10	10	10	11	11

x	12	13	14	15	16	17	18	19	20	21
$f(x)$	11	12	12	12	13	13	13	14	14	14

Let S denote the set {1, 2.3…., 20, 21}, and let S1 = f(S),S2 = f(S1).S3 = f(S2)

….. We now have:

S1 = {8,9,10,11,12,13,14},

S2 = {10,11,12},

S3 = {11}.

Thus in just 3 iterations, f maps S to {11}, a single-element set! Thus the chosen card necessarily ends up at position number 11 after 3 iterations, and the trick becomes transparent.

14.2.4

● The procedure we shall use to solve this problem is essentially the same as that used in the text to tabulate the cycles of the SCB and SFT functions. Denote by f the function that computes the sum of the fifth powers of the digits of the positive integer x, and let f act on sets of integers as well as single integers. Next, denote by g the function that acts thus: for any set of positive integers X,

$$g(X) = f(X) \cap X.$$

We first look for an integer N such that for all $x < N$ we have $f(x) < N$. After some experimentation, it turns out that N = 300000 works. (Proof: In the interval from 1 till 300000 the integer with the largest f-value is 299999, and $f(299999) = 295277 < 29999$.)

Let S denote the set {1,2, 3, …,300000}, and let g operate iteratively on S. As earlier, what we need to find is a *g*-invariant subset of S. As S is a rather large set, in order to make progress we need access to powerful computing machinery. The author's facilities proved unequal to the task and therefore did not permit him to carry out a complete analysis. Thus only partial results are reported below.

FIXED POINTS: We find that after 10 iterations on {1,2,3,…,30000 }, we reach the set

$$T = \{0,1, 4150,4151 \}$$

which is easily checked to be *g*-invariant; indeed, each element of T is a fixed point of f.

For instance, f(4150) = 1024 + 1 + 3125 + 0 = 4150. Thus f has at least these four fixed points.

We also obtain the following cycles of f, including a cycle of length 28 (!) .

2-CYCLE:⟨58618,76438⟩.

10-CYCLE: ⟨18107, 49577, 96812, 99626, 133682, 41063, 9044, 61097, 83633, 41273⟩.

22-CYCLE:
⟨9045,63198,99837,167916,91410,60075,27708,66414,17601,24585,40074,1885

28-CYCLE:
⟨244,2080,32800,33043,1753,20176,24616,16609,74602,25639,70225,19996,18

The reader is invited to unravel the full cycle structure of f.

● We follow the approach of Problem 11.3.1 yet again. Let f denote the function under consideration (thus f(23) = 12,f(35) = 24, and so on) and let f act on sets as well; for instance,

$$f(\{10,11,12\}) = \{2,4,6\}, f(25,30,35\}\ ) = \{18,4,24\}.$$

Next, we define a function g that acts on sets of positive integers thus: for any such set X, let g(X) = f(X) ⋂ X. For instance, we have

$$g(\{1,2,3,4,5\}) = \{2,3,4,5,6\} \bigcap \{1,2,3,4,5\} = \{2,3,4,5\}.$$

We now let g act iteratively on a large enough set, say the set

$$S = \{1,2,3,\ldots,9999\},$$

that is, we study the orbit ⟨S;g⟩. It turns out that after only 10 iterations or so we reach an invariant set, namely the set

$$T = \{2,3,4,5,6,7,8,9,10,18\}.$$

From this *g*-invariant set, we extract one fixed point and one 9-cycle:

$$\langle 18\rangle;\langle 2,3,4,5,6,7,8,9,10\rangle.$$

The numerical evidence suggests that these are the only cycles of f. Perhaps the reader could. investigate what happens if we start with a larger set S, say the set of integers till 99999

14.3.1 This is clear; if a,b,c,d are all positive, then $|a - b| \leq \max\{a,b\}, |b - c| \leq \max\{b,c\}$, and so on. Therefore

$$\max\{e,f,g,h\} \leq \max\{a,b,c,d\}.$$

14.3.2 This is shown by examining all possible cases. Write 'E' for 'even number' and '*O*' for 'odd number'; then it is easy to see that:

$$(O,O,O,O) \mapsto (E,E.,E,E),$$

$$(E,O,E,O) \mapsto (O,O,O,O),$$

$$(E,O,E,E) \mapsto (O,O,E,E),$$

$$(O,O,E,E) \mapsto (E,O,E,O),$$

and so on. Tracing out the orbits, we see that we always reach (E,E,E,E) within at most four steps.

14.3.3 This is the clear, because height (e,f,g,h) = 1 2(

$$\text{height } (2e,2f,2g,2h)) \leq \text{height } (a,b,c,d).$$

14.3.4 The hint given in the text should suffice for this.

14.3.5

i. $f(0.9) = 0$
ii. $f(1.5) = 4$ (attained by the quadruples (1, 0,0,0) and (1, 1, 1,0)).
iii. $f(3) = 4$ (attained by several quadruples, e.g., (2,0,0,0) ,(2,2,1,0), and so on).
iv. $f(4) = 6$ (attained by (0, 0, 1, 3) and (0,2,3,3)).

Further experimentation yields $f(5) = 7$, attained by (0,1,2,4) and (0,2,3,4).

14.3.6 All quadruples considered during the computation of f(x) are also considered during the computation of f(y), because $x \leq y$, so we must have $f(x) \leq f(y)$.

14.3.7 As mentioned earlier, if the largest entry is less than 1, then all the entries must be 0, which means that the life is 0.

14.3.8 This is obvious!

14.3.9 As shown earlier, starting from any quadruple (a,b,c,d), within four steps at most we reach a quadruple all of whose entries are even numbers, say (2e,2f,2g,2h). The life of (2e,2f,2g,2h) is equal to that of (e,f,g,h), so we see that

$$\text{life}((a,b,c,d)) \leq 1\text{ife}((e,f,g,h)) + 4.$$

Now if the quantities a,b,c,d are all less than x, then the quantities 2e,2f,2g,2h too are less than x, and therefore the quantities e,f,g,h are all less than $x/2$. This shows that $f(x) \leq f(x/2) + 4$, as required.

14.3.10 Using the inequality $f(x) \leq f(x/2) + 4$ over again, we see that:

$$f(x) \leq f\,x\,4 + 4 \leq f\,x\,8 + 8 \leq f\,x\,16 + 12 \leq \ldots\ldots.$$

The quantities $f(x/2), f(x/4), f(x/8), \ldots$ shrink to zero rapidly, and the result follows. Observe that this argument also proves the following result: *If* $x < 2n$, *then* $f(x) < 4n$.

14.3.11 The argument fails at a crucial point: it is not true that after every so many steps (whatever the number), each number in the triple is even (or a multiple of three, or whatever). So the possibility of reducing the height of the triple does not exist, as it did in the case of quadruples.

14.3.12 The answer is: *yes*. For, we can express all the fractions in terms of a common denominator, and then simply remove the denominator from the picture. Consider the following example:

$$2\ 3,3\ 7,\ 5\ 12,\ 3\ 19 \mapsto 5\ 21,\ 1\ 84,\ 59\ 228,29\ 57.$$

Since the least common multiple of 3, 7, 12 and 19 is $7 \times 12 \times 19 = 1596$, and $2/3 = 1024/1596$, $3/7 = 684/1596, 5/12 = 665/1596, 3/19 = 252/1596$, the computation por-trayed above is essentially the same as:

$$(1024, 684, 665, 252) \mapsto (340,19,413,772).$$

The earlier argument thus applies, and we conclude that each quadruple of rational numbers has finite life.

14.3.13 The answer to this is: no. That is, it is possible to find quadruples of numbers, at least one being irrational, whose life is not finite. One such quadruple is constructed in the following manner. Let α be the positive real number that satisfies the equation

$$\alpha 3 = \alpha 2 + \alpha + 1.$$

Then the quadruple $(1,\alpha,\alpha 2,\alpha 3)$ has infinite life. (It can be shown that α is irrational; its value is approximately 1.83928….) To see why, we examine its orbit:

$$(1,\alpha,\alpha 2,\alpha 3)\mapsto(\alpha - 1,\alpha 2 - \alpha,\alpha 3 - \alpha 2,\alpha 3 - 1).$$

Now $\alpha 3 - 1 = (\alpha - 1)(\alpha 2 + \alpha + 1)$, so the factor $(\alpha - 1)$, being common to all four numbers in the quadruple, can be factored out. By the choice of α, we have the relation $\alpha 3 = \alpha 2 + \alpha + 1$, so the quadruple can be written in the form $(\alpha - 1) \times (1,\alpha,\alpha 2,\alpha 3)$. Let L denote the life of $(1,\alpha,\alpha 2,\alpha 3)$. The relation just found states that $L = L + 1$, which is an impossible equation (it cannot be satisfied by any finite number). Conclusion: L is infinite!

14.3.14 This question has already been answered (see Problem 6.2.7). The crucial point is that the remainders that can arise in the division $a \div b$ form a *finite* set: $\{0,1,2,\ldots,b - 1\}$. Therefore some remainder will necessarily repeat after some point, and once this happens the decimal recurs.

15.1.1

i. To solve the equation $x5 - x - 1000=0$, we write it in the form $x = (x + 1000)1/5$.

   We now let f be the function given by

   $$f(x) = (x + 1000)1/5$$

   and search for a fixed point of f. A computer search provides the following orbit, using the seed $x = 0$.

   $$\langle 0,3.9811,3.9842,3.9842,3.9842,\ldots\rangle.$$

   Thus we have found a fixed point (≈ 3.984). This is one solution to the given equation.

ii. We write the equation in the form $x = (x4+ x + 10000)1/7$. . Let g be the function given by

   $$g(x) = (x4 + x + 1000)\ 1/7.$$

   Using the seed $x = 0$, we obtain the orbit shown below:

⟨0,3.7276,3.738,3.7381,3.7381,...,⟩.

We have found a solution (≈ 3.738).

15.1.2 A particularly good choice for the iteration is

x↦x - y 4x3 - 3x2 - 2κ → 1.

Line 80 in this case would read:

80 x = x -y⁄(4*x⋀3 - 3*x⋀2 - 2*x -1)

15.1.3 We write the equation as x = (x1/3+100)2 and search for a fixed point of the function h, where

h(x) = (x1⁄3 + 100)2

The seed x = 0 gives the following orbit, with numbers rounded off to the closest integer:

⟨0,10000,14773,15509,15610,15623,15625,15625,15625,15625,…⟩.

We have obtained a fixed point of h, namely.the number 15625. Note that this is an *exact* solution to our problem. (Proof: 15625 = 125,156253 = 25,125 - 25 = 100.)

15.1.4 We write the equation as x = (x + 1)x⁄(x+1) and search for a fixed point of the function k, where

k(x) = (x + 1)x⁄(x+1).

The seed has to be chosen with care; x = 0 would not be a good choice. We use the seed x = 2. The result is as follows. ⟨2,2.08,2.14,2.18,2.21,2.23,2.25,2.26,2.27,2.275,2.28,2.283,2.286,2.288,2.289,2.2 …⟩.

There seems to be a solution close to 2.29., we run a second iteration using the seed x = 2.3. Here is the result: ⟨2.3,2.298,2.297,2.2’96,2.295,2.2947,2.2943,2.294,2.2938,2.2936,2.2935,2.2934, … ⟩ .

A good approximation to the limit, obtained by using more efficient methods, is 2.2931662.

★ ★ ★ ★ ★ ★ ★ ★ ★ ★ ★★★ ★ ★ ★ ★ ★★

We leave the solutions of **17.5.1, 17.5.2, 17.5.3** and **17.5.4** to the reader, as the proofs of the three theorems are highly similar to the proofs of Theorems 17.1 and 17.2.